Types and Terms Translated: Unrestricted Resources in Encoding Functions as Processes

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Abstract

Type-preserving translations are effective rigorous tools in the study of core programming calculi. In this paper, we develop a new typed translation that connects sequential and concurrent calculi; it is governed by type systems that control *resource consumption*. Our main contribution is the source language, a new resource λ-calculus with non-determinism and failures, dubbed $u\lambda^{\lightning}_{\oplus}$. In $u\lambda^{\lightning}_{\oplus}$, resources are split into linear and unrestricted; failures are explicit and arise from this distinction. We define a type system based on intersection types to control resources and fail-prone computation. The target language is $\mathsf{s}\pi$, an existing session-typed π-calculus that results from a Curry-Howard correspondence between linear logic and session types. Our typed translation subsumes our prior work; interestingly, it treats unrestricted resources in $u\lambda^{\lightning}_{\oplus}$ as client-server session behaviours in $\mathsf{s}\pi$.

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1 Introduction

Context *Type-preserving translations* are effective rigorous tools in the study of core programming calculi. They can be seen as an abstract counterpart to the type-preserving compilers that enable key optimisations in the implementation of programming languages. The goal of this paper is to develop a new typed translation that connects sequential and concurrent calculi, and is governed by type systems that control *resource consumption*.

A central idea in the resource λ-calculus is to consider that in an application $M\,N$ the argument N is a *resource* of possibly limited availability. This generalisation of the λ-calculus triggers many fascinating questions, such as typability, solvability, expressiveness power, etc., which have been studied in different settings (see, e.g., [1, 3, 16, 7]). In established resource λ-calculi, such as those by Boudol [1] and by Pagani and Ronchi della Rocca [16], a more general form of application is considered: a term can be applied to a bag of resources $B = \wr N_1 \int \cdot \ldots \cdot \wr N_k \int$, where $N_1, \ldots, N_k$ denote terms; then, an application $M\ B$ must take into account that each N_i may be reusable or not. Thus, non-determinism is natural in resource λ-calculi, because a term has now multiple ways of consuming resources from the bag. This bears a strong resemblance with process calculi such as the π-calculus [15], in which concurrent interactions are intrinsically non-deterministic.

There are different flavors of non-determinism. Over two decades ago, Boudol and Laneve [2, 3] explored connections between a resource λ-calculus and the π-calculus. In their setting, an application $M\ B$ would branch, i.e., M could consume a resource N_j in B (with $j \in \{1, \ldots k\}$) and discard the other $k-1$ resources in a non-confluent manner; this is what we call a *collapsing* approach to non-determinism. On a different direction, Pagani and Ronchi della Rocca [16] proposed λ^r, a resource λ-calculus that implements *non-collapsing* non-determinism, whereby all the possible alternatives for resource consumption are retained together in a sum, ensuring confluence. They investigated typability and characterisations of solvability in λ^r, but no connection with the π-calculus was established. In an attempt to address this gap, our previous work [17] identified $\lambda^{\lightning}_{\oplus}$, a resource λ-calculus with non-collapsing non-determinism, explicit failure, and *linear* resources (to be used exactly once), and developed a correct typed translation into a session typed π-calculus [5]. The calculus $\lambda^{\lightning}_{\oplus}$, however, does not include *unrestricted* resources (to be used zero or many times).

This Paper Here we introduce a new λ-calculus, dubbed $u\lambda^{\lightning}_{\oplus}$, its intersection type system, and its translation into session-typed processes. Our motivation is twofold: to elucidate the status of unrestricted resources in a functional setting with non-collapsing non-determinism, and to characterise unrestricted resources within a translation of functions into processes. Unlike its predecessors, $u\lambda^{\lightning}_{\oplus}$ distinguishes between linear and unrestricted resources. This distinction determines the semantics of terms and especially the deadlocks (*failures*) that arise due to mismatches in resources. This way, $u\lambda^{\lightning}_{\oplus}$ subsumes $\lambda^{\lightning}_{\oplus}$, which is purely linear and cannot express failures related to unrestricted resources.

Distinguishing linear and unrestricted resources is not a new insight. This idea goes back to Boudol's λ-calculus with multiplicities [1], where arguments can be tagged as unrestricted. What is new about $u\lambda^{\lightning}_{\oplus}$ is that the distinction between linear and unrestricted resources leads to two main differences. First, occurrences of a variable can be linear or unrestricted, depending on the kind of resources they should be substituted with. This way, e.g., a linear occurrence of variable must be substituted with a linear resource. In $u\lambda^{\lightning}_{\oplus}$, a variable can have linear and unrestricted occurrences in the same term. (Notice that we use the adjective 'linear' in connection to resources used exactly once, and not to the number of occurrences of a variable in a term.) Second, failures depend on the nature of the involved resource(s). In $u\lambda^{\lightning}_{\oplus}$, a linear failure arises from a mismatch between required and available (linear) resources; an unrestricted failure arises when a specific (unrestricted) resource is not available.

Accordingly, the syntax of $u\lambda^{\lightning}_{\oplus}$ incorporates linear and unrestricted resources, enabling their consistent separation, within non-collapsing non-determinism. The calculus allows for linear and unrestricted occurrences of variables, as just discussed; bags comprise two separate zones, linear and unrestricted; and the *failure term* $\mathtt{fail}^{x_1,\cdots,x_n}$ explicitly mentions the linear variables $x_1, \ldots, x_n$. The (lazy) reduction semantics of $u\lambda^{\lightning}_{\oplus}$ includes two different rules for "fetching" terms from bags, and for consistently handling the failure term.

We equip $u\lambda^{\lightning}_{\oplus}$ with non-idempotent intersection types, extending the approach in [17]: in $u\lambda^{\lightning}_{\oplus}$, intersection types account for more than resource multiplicity, since the elements of the unrestricted bag can have different types. Using intersection types, we define a class of *well-formed* $u\lambda^{\lightning}_{\oplus}$ expressions, which includes terms that correctly consume resources but also terms that may reduce to the failure term. Well-formed expressions thus subsume the *well-typed* expressions that can be defined in a sub-language of $u\lambda^{\lightning}_{\oplus}$ without the failure term.

The calculus $u\lambda^{\lightning}_{\oplus}$ can express terms whose dynamic behaviour is not captured by prior works. This way, e.g., the identity function $\mathbf{I}$ admits two formulations, depending on whether the variable occurrence is linear or unrestricted. One can have $\lambda x.x$, as usual, but also the unrestricted variant $\lambda x.x[i]$, where '$[i]$' is an index annotation (similar to a qualifier or a tag), which indicates that x should be replaced by the i-th element of the unrestricted zone of the bag. The behaviour of these functions will depend on the bags that are provided as their arguments. Similarly, we can express variants of $\Delta = \lambda x.xx$ and $\Omega = \Delta\,\Delta$ whose behaviours again depend on linear or unrestricted occurrences of variables and bags. Consider the term $\Delta_7 = \lambda x.(x[1](1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!))$, where we use '$\star$' to separate linear and unrestricted resources in the bag, and '$\diamond$' denotes concatenation of unrestricted resources. Term Δ_7 is an abstraction on x of an application of an unrestricted occurrence of x, which aims to consume the first component of an unrestricted bag, to a bag with an empty linear zone (denoted 1) and an unrestricted zone with resources $\wr x[1] \int^!$ and $\wr x[2] \int^!$. The self-application $\Delta_7\Delta_7$ produces a non-terminating behaviour and yet Δ_7 itself is well-formed (see Example 16).

Both $u\lambda^{\lightning}_{\oplus}$ and $\lambda^{\lightning}_{\oplus}$ are *logically motivated* resource λ-calculi, in the following sense: their design has been strongly influenced by $\mathsf{s}\pi$, a typed π-calculus resulting from the Curry-Howard correspondence between linear logic and session types in [5], where proofs correspond to processes and cut elimination to process communication. As demonstrated in [5], providing primitive support for explicit failures is key to expressing many useful programming idioms (such as exceptions); this insight is a leading motivation in our design for $u\lambda^{\lightning}_{\oplus}$.

To attest to the logical underpinnings of $u\lambda^{\lightning}_{\oplus}$, we develop a typed translation (or *encoding*) of $u\lambda^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$ and establish its correctness with respect to well-established criteria [9, 14]. As in [17], we encode $\lambda^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$ by relying on an intermediate language with *sharing* constructs [10, 8, 13]. A key idea in encoding $u\lambda^{\lightning}_{\oplus}$ is to codify the behaviour of unrestricted occurrences of a variable and their corresponding resources in the bag as *client-server connections*, leveraging the copying semantics for the exponential "$!A$" induced by the Curry-Howard correspondence. This typed encoding into $\mathsf{s}\pi$ justifies the semantics of $u\lambda^{\lightning}_{\oplus}$ in terms of precise session protocols (i.e., linear logic propositions, because of the correspondence).

In summary, the **main contributions** of this paper are: (1) The resource calculus $u\lambda^{\lightning}_{\oplus}$ of linear and unrestricted resources, and its associated intersection type system. (2) A typed encoding of $u\lambda^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$, which connects well-formed expressions (disciplined by intersection types) and well-typed concurrent processes (disciplined by session types, under the Curry-Howard correspondence with linear logic), subsuming the results in [17].

Additional Material The appendices contain omitted material. App. A collects technical details on $u\lambda^{\lightning}_{\oplus}$. App. B details the proof of subject reduction for well-formed $u\lambda^{\lightning}_{\oplus}$ expressions. App. C–App. F collect omitted definitions and proofs for our encoding of $u\lambda^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$.

2 $u\lambda^{\lightning}_{\oplus}$: Unrestricted Resources, Non-Determinism, and Failure

Syntax. We shall use $x, y, \ldots$ to range over *variables*, and $i, j \ldots$, as positive integers, to range over *indices*. Variable occurrences will be *annotated* to distinguish the kind of resource they should be substituted with (linear or unrestricted). With a slight abuse of terminology, we may write 'linear variable' and 'unrestricted variable' to refer to linear and unrestricted occurrences of a variable. As we will see, a variable's annotation will be inconsequential for binding purposes. We write $\widetilde{x}$ to abbreviate $x_1, \ldots, x_n$, for $n \geq 1$ and each x_i distinct.

▶ **Definition 1** ($u\widehat{\lambda}^{\lightning}_{\oplus}$). *We define* terms *($M, N$),* bags *($A, B$), and* expressions *($\mathbb{M}, \mathbb{N}$) as:*

(Annotations)	$[*] ::= [i] \mid [\ell] \qquad i \in \mathbb{N}$
(Terms)	$M, N ::= x[*] \mid \lambda x.M \mid (M\ B) \mid M\langle\!\langle B/x \rangle\!\rangle \mid \mathtt{fail}^{\widetilde{x}}$
(Linear Bags)	$C, D ::= \mathtt{1} \mid \wr M \int \cdot C$
(Unrestricted Bags)	$U, V ::= \mathtt{1}^! \mid \wr M \int^! \mid U \diamond V$
(Bags)	$A, B ::= C \star U$
(Expressions)	$\mathbb{M}, \mathbb{N} ::= M \mid \mathbb{M} + \mathbb{N}$

To lighten up notation, we shall omit the annotation for linear variables. This way, e.g., we write $(\lambda x.x)B$ rather than $(\lambda x.x[\ell])B$.

Definition 1 introduces three syntactic categories: *terms* (in functional position); *bags* (multisets of resources, in argument position), and *expressions*, which are finite formal sums that denote possible results of a computation. Below we describe each category in details.

- Terms (unary expressions):
  - Variables: We write $x[\ell]$ to denote a *linear* occurrence of x, i.e, an occurrence that can only be substituted for linear resources. Similarly, $x[i]$ denotes an *unrestricted* occurrence of x, i.e., an occurrence that can only be substituted for a resource located at the i-th position of an unrestricted bag.
  - Abstractions $\lambda x.M$ of a variable x in a term M, which may have contain linear or unrestricted occurrences of x. This way, e.g., $\lambda x.x$ and $\lambda x.x[i]$ are linear and unrestricted versions of the identity function. Notice that the scope of x is M, as usual, and that $\lambda x.(\cdot)$ binds both linear and unrestricted occurrences of x.
  - Applications of a term M to a bag B (written $M\ B$) and the explicit substitution of a bag B for a variable x (written $\langle\!\langle B/x \rangle\!\rangle$) are as expected (cf. [1, 3]). Notice that in $M\langle\!\langle B/x \rangle\!\rangle$ the occurrences of x in M, linear and unrestricted, are bound. Some conditions apply to B: this will be evident later on, after we define our operational semantics (cf. Fig. 1).
  - The failure term $\mathtt{fail}^{\widetilde{x}}$ denotes a term that will result from a reduction in which there is a lack or excess of resources, where $\widetilde{x}$ denotes a multiset of free linear variables that are encapsulated within failure.
- A bag B is defined as $C \star U$: the concatenation of a bag of linear resources C with a bag (actually, a list) of unrestricted resources U. We write $\wr M \int$ to denote the linear bag that encloses term M, and use $\wr M \int^!$ in the unrestricted case.
  - Linear bags ($C, D, \ldots$) are multisets of terms. The empty linear bag is denoted $\mathtt{1}$. We write $C_1 \cdot C_2$ to denote the concatenation of C_1 and C_2; this is a commutative and associative operation, where $\mathtt{1}$ is the identity.
  - Unrestricted bags ($U, V, \ldots$) are ordered lists of terms. The empty unrestricted bag is denoted as $\mathtt{1}^!$. The concatenation of U_1 and U_2 is denoted by $U_1 \diamond U_2$; this operation is associative but not commutative. Given $i \geq 1$, we write U_i to denote the i-th element of the unrestricted (ordered) bag U.
- Expressions are sums of terms, denoted as $\sum_i^n N_i$, where $n > 0$. Sums are associative and commutative; reordering of the terms in a sum is performed silently.

▶ **Example 2.** Consider the term $M := \lambda x.(x[1] \wr x \int \star \wr y[1] \int^!)$, which has linear and unrestricted occurrences of the same variable. This is an abstraction of an application that contains two bound occurrences of x (one unrestricted with index 1, and one linear) and one free unrestricted occurrence of $y[1]$, occurring in an unrestricted bag. As we will see, in $M\ (C \star U)$, the unrestricted occurrence '$x[1]$' should be replaced by the first element of U.

The salient features of $u\lambda^{\lightning}_{\oplus}$—the explicit construct for failure, the index annotations on unrestricted variables, the ordering of unrestricted bags—are *design choices* that will be responsible for interesting behaviours, as the following examples illustrate.

▶ **Example 3.** As already mentioned, $u\lambda^{\lightning}_{\oplus}$ admits different variants of the usual λ-term $\mathbf{I} = \lambda x.x$. We could have one in which x is a linear variable (i.e., $\lambda x.x$), but also several possibilities if x is unrestricted (i.e., $\lambda x.x[i]$, for some positive integer i). Interestingly, because $u\lambda^{\lightning}_{\oplus}$ supports failures, non-determinism, and the consumption of arbitrary terms of the unrestricted bag, these two variants of $\mathbf{I}$ can have behaviours that may differ from the usual interpretation of $\mathbf{I}$. In Example 9 we will show that the six terms below give different behaviours:

- $M_1 = (\lambda x.x)(\wr N \int \star U)$
- $M_2 = (\lambda x.x)(\wr N_1 \int \cdot \wr N_2 \int \star U)$
- $M_3 = (\lambda x.x[1])(\wr N \int \star \mathtt{1}^!)$
- $M_4 = (\lambda x.x[1])(\mathtt{1} \star \wr N \int^! \diamond U)$
- $M_5 = (\lambda x.x[1])(\mathtt{1} \star \mathtt{1}^! \diamond U)$
- $M_6 = (\lambda x.x[i])(C \diamond U)$

We will see that M_1, M_4, M_6 reduce without failures, whereas M_2, M_3, M_5 reduce to failure.

▶ **Example 4.** Similarly, $u\lambda^{\lightning}_{\oplus}$ allows for several forms of the standard λ-terms such as $\Delta := \lambda x.xx$ and $\Omega := \Delta\Delta$, depending on whether the variable x is linear or unrestricted:

1. $\Delta_1 := \lambda x.(x(\wr x \int \star \mathtt{1}^!))$ consists of an abstraction of a linear occurrence of x applied to a linear bag containing another linear occurrence of x. There are two forms of self-applications of Δ_1, namely: $\Delta_1(\wr \Delta_1 \int \star \mathtt{1}^!)$ and $\Delta_1(\mathtt{1} \star \wr \Delta_1 \int^!)$.
2. $\Delta_4 := \lambda x.(x[1](\wr x \int \star \mathtt{1}^!))$ consists of an unrestricted occurrence of x applied to a linear bag (containing a linear occurrence of x) that is composed with an empty unrestricted bag. Similarly, there are two self-applications of Δ_4, namely: $\Delta_4(\wr \Delta_4 \int \star \mathtt{1}^!)$ and $\Delta_4(\mathtt{1} \star \wr \Delta_4 \int^!)$.
3. We show applications of an unrestricted variable occurrence ($x[2]$ or $x[1]$) applied to an empty linear bag composed with a non-empty unrestricted bag (of size two):
   - $\Delta_3 = \lambda x.(x[1](\mathtt{1} \star \wr x[1] \int^! \diamond \wr x[1] \int^!))$
   - $\Delta_5 := \lambda x.(x[2](\mathtt{1} \star \wr x[1] \int^! \diamond \wr x[2] \int^!))$
   - $\Delta_6 := \lambda x.(x[1](\mathtt{1} \star \wr x[1] \int^! \diamond \wr x[2] \int^!))$
   - $\Delta_7 := \lambda x.(x[2](\mathtt{1} \star \wr x[1] \int^! \diamond \wr x[1] \int^!))$

   Applications between these terms express behaviour, similar to a lazy evaluation of Ω:
   - $\Omega_5 := \Delta_5(\mathtt{1} \star \wr \Delta_5 \int^! \diamond \wr \Delta_5 \int^!)$
   - $\Omega_{5,6} := \Delta_5(\mathtt{1} \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^!)$
   - $\Omega_{6,5} := \Delta_6(\mathtt{1} \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^!)$
   - $\Omega_7 := \Delta_7(\mathtt{1} \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^!)$

The behaviour of these terms will be made explicit later on (see Examples 11 and 12).

Semantics. The semantics of $u\lambda^{\lightning}_{\oplus}$ captures that linear resources can be used only once, and that unrestricted resources can be used *ad libitum.* Thus, the evaluation of a function applied to a multiset of linear resources produces different possible behaviours, depending on the way these resources are substituted for the linear variables. This induces non-determinism, which we formalise using a *non-collapsing* approach, in which expressions keep all the different possibilities open, and do not commit to one of them. This is in contrast to *collapsing* non-determinism, in which selecting one alternative discards the rest.

We define a reduction relation $\longrightarrow$, which operates lazily on expressions. Informally, a β-reduction induces an explicit substitution of a bag $B = C \star U$ for a variable x, denoted $\langle\!\langle B/x \rangle\!\rangle$, in a term M. This explicit substitution is then expanded depending on whether the head of M has a linear or an unrestricted variable. Accordingly, in $u\lambda^{\lightning}_{\oplus}$ there are *two sources of failure*: one concerns mismatches on linear resources (required vs available resources); the other concerns the unavailability of a required unrestricted resource (an empty bag $\mathtt{1}^!$).

To formalise reduction, we require a few auxiliary notions.

▶ **Definition 5.** *The multiset of free linear variables of* $\mathbb{M}$*, denoted* $\mathsf{mlfv}(\mathbb{M})$*, is defined below. We denote by* $[x]$ *the multiset containing the linear variable* x *and* $[x_1,\ldots,x_n]$ *denotes the multiset containing* $x_1,\ldots,x_n$*. We write* $\widetilde{x}\uplus\widetilde{y}$ *to denote the multiset union of* $\widetilde{x}$*, and* $\widetilde{y}$ *and* $\widetilde{x}\setminus y$ *to express that every occurrence of* y *is removed from* $\widetilde{x}$*.*

$$
\begin{aligned}
\mathsf{mlfv}(x) &= [x] & \mathsf{mlfv}(x[i]) &= \mathsf{mlfv}(\mathtt{1}) = \emptyset\\
\mathsf{mlfv}(C\star U) &= \mathsf{mlfv}(C) & \mathsf{mlfv}(M\ B) &= \mathsf{mlfv}(M)\uplus\mathsf{mlfv}(B)\\
\mathsf{mlfv}(\wr M \int) &= \mathsf{mlfv}(M) & \mathsf{mlfv}(\lambda x.M) &= \mathsf{mlfv}(M)\setminus\{x\}\\
\mathsf{mlfv}(M\langle\!\langle B/x\rangle\!\rangle) &= (\mathsf{mlfv}(M)\setminus\{x\})\uplus\mathsf{mlfv}(B) & \mathsf{mlfv}(\wr M\int\cdot C) &= \mathsf{mlfv}(M)\uplus\mathsf{mlfv}(C)\\
\mathsf{mlfv}(\mathbb{M}+\mathbb{N}) &= \mathsf{mlfv}(\mathbb{M})\uplus\mathsf{mlfv}(\mathbb{N}) & \mathsf{mlfv}(\mathtt{fail}^{x_1,\cdots,x_n}) &= [x_1,\ldots,x_n]
\end{aligned}
$$

A term M *(resp. expression* $\mathbb{M}$*) is called* linearly closed *if* $\mathsf{mlfv}(M)=\emptyset$ *(resp.* $\mathsf{mlfv}(\mathbb{M})=\emptyset$*).*

▶ **Notation 1.** *We shall use the following notations.*

- $N\in\mathbb{M}$ *means that* N *occurs in the sum* $\mathbb{M}$*. Also, we write* $N_i\in C$ *to denote that* N_i *occurs in the linear bag* C*, and* $C\setminus N_i$ *to denote the linear bag obtained by removing one occurrence of* N_i *from* C*.*
- $\#(x,M)$ *denotes the number of (free) linear occurrences of* x *in* M*. Also,* $\#(x,\widetilde{y})$ *denotes the number of occurrences of* x *in the multiset* $\widetilde{y}$*.*
- $\mathsf{PER}(C)$ *is the set of all permutations of a linear bag* C *and* $C_i(n)$ *denotes the* n*-th term in the (permuted)* C_i*.*
- $\mathsf{size}(C)$ *denotes the number of terms in a linear bag* C*. That is,* $\mathsf{size}(\mathtt{1})=0$ *and* $\mathsf{size}(\wr M\int\cdot C)=1+\mathsf{size}(C)$*. Given a bag* $B=C\star U$*, we define* $\mathsf{size}(B)$ *as* $\mathsf{size}(C)$*.*

▶ **Definition 6** (Head)**.** *Given a term* M*, we define* $\mathsf{head}(M)$ *inductively as:*

$$
\mathsf{head}(x)=x \qquad \mathsf{head}(M\ B)=\mathsf{head}(M) \qquad \mathsf{head}(\lambda x.M)=\lambda x.M
$$

$$
\mathsf{head}(x[i])=x[i] \quad \mathsf{head}(\mathtt{fail}^{\widetilde{x}})=\mathtt{fail}^{\widetilde{x}} \qquad \mathsf{head}(M\langle\!\langle B/x\rangle\!\rangle)=\begin{cases}\mathsf{head}(M) & \text{if } \#(x,M)=\mathsf{size}(B)\\ \mathtt{fail}^{\emptyset} & \text{otherwise}\end{cases}
$$

▶ **Definition 7** (Head Substitution)**.** *Let* M *be a term such that* $\mathsf{head}(M)=x$*. The* head substitution *of a term* N *for* x *in* M*, denoted* $M\{\!|N/x|\!\}$*, is inductively defined as follows (where* $x\neq y$*):*

$$
x\{\!|N/x|\!\}=N \quad (M\ B)\{\!|N/x|\!\}=(M\{\!|N/x|\!\})\ B \quad (M\ \langle\!\langle B/y\rangle\!\rangle)\{\!|N/x|\!\}=(M\{\!|N/x|\!\})\ \langle\!\langle B/y\rangle\!\rangle
$$

When $\mathsf{head}(M)=x[i]$, the head substitution $M\{\!|N/x[i]|\!\}$ works as expected: $x[i]\{\!|N/x[i]|\!\}=N$ as the base case of the definition. Finally, we define contexts for terms and expressions:

▶ **Definition 8** (Evaluation Contexts)**.** *Contexts for terms (CTerm) and expressions (CExpr) are defined by the following grammar:*

$$
\textit{(CTerm)}\quad C[\cdot],C'[\cdot] ::= ([\cdot])B \mid ([\cdot])\langle\!\langle B/x\rangle\!\rangle \qquad \textit{(CExpr)}\quad D[\cdot],D'[\cdot] ::= M+[\cdot]
$$

Reduction is defined by the rules in Fig. 1. Rule $[\mathtt{R}:\mathtt{Beta}]$ induces explicit substitutions. Resource consumption is implemented by two fetch rules, which open up explicit substitutions:

- Rule $[\mathtt{R}:\mathtt{Fetch}^{\ell}]$, the *linear fetch*, ensures that the number of required resources matches the size of the linear bag C. It induces a sum of terms with head substitutions, each denoting the partial evaluation of an element from C. Thus, the size of C determines the summands in the resulting expression.

$$[\mathtt{R} : \mathtt{Beta}]\ \frac{}{(\lambda x.M)B \longrightarrow M\langle\langle B/x\rangle\rangle}$$

$$[\mathtt{R} : \mathtt{Fetch}^{\ell}]\ \frac{\mathsf{head}(M) = x \qquad C = \wr N_1 \int \cdot \cdots \cdot \wr N_k \int\ ,\ k \geq 1 \qquad \#(x, M) = k}{M\langle\langle C \star U/x\rangle\rangle \longrightarrow M\{\!| N_1/x |\!\}\langle\langle (C \setminus N_1) \star U/x\rangle\rangle + \cdots + M\{\!| N_k/x |\!\}\langle\langle (C \setminus N_k) \star U/x\rangle\rangle}$$

$$[\mathtt{R} : \mathtt{Fetch}^{!}]\ \frac{\mathsf{head}(M) = x[i] \quad \#(x, M) = \mathsf{size}(C) \qquad U_i = \wr N \int^{!}}{M\ \langle\langle C \star U/x\rangle\rangle \longrightarrow M\{\!| N/x[i] |\!\}\langle\langle C \star U/x\rangle\rangle}$$

$$[\mathtt{R} : \mathtt{Fail}^{\ell}]\ \frac{\#(x, M) \neq \mathsf{size}(C) \quad \widetilde{y} = (\mathsf{mlfv}(M) \setminus x) \uplus \mathsf{mlfv}(C)}{M\langle\langle C \star U/x\rangle\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}}}$$

$$[\mathtt{R} : \mathtt{Fail}^{!}]\ \frac{\#(x, M) = \mathsf{size}(C) \qquad U_i = 1^{!} \quad \mathsf{head}(M) = x[i]}{M\langle\langle C \star U/x\rangle\rangle \longrightarrow M\{\!| \mathtt{fail}^{\emptyset}/x[i] |\!\}\langle\langle C \star U/x\rangle\rangle}$$

$$[\mathtt{R} : \mathtt{Cons}_1]\ \frac{\widetilde{y} = \mathsf{mlfv}(C)}{(\mathtt{fail}^{\widetilde{x}})\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}}}$$

$$[\mathtt{R} : \mathtt{Cons}_2]\ \frac{\#(z, \widetilde{x}) = \mathsf{size}(C)\ \widetilde{y} = \mathsf{mlfv}(C)}{\mathtt{fail}^{\widetilde{x}}\ \langle\langle C \star U/z\rangle\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{(\widetilde{x} \setminus z) \uplus \widetilde{y}}}$$

$$[\mathtt{R} : \mathtt{ECont}]\ \frac{\mathbb{M} \longrightarrow \mathbb{M}'}{D[\mathbb{M}] \longrightarrow D[\mathbb{M}']} \qquad [\mathtt{R} : \mathtt{TCont}]\ \frac{M \longrightarrow \sum_{i=1}^{k} M_i'}{C[M] \longrightarrow \sum_{i=1}^{k} C[M_i']}$$

Figure 1 Reduction rules for $u\lambda^{\lightning}_{\oplus}$.

- Rule $[\mathtt{R} : \mathtt{Fetch}^{!}]$, the *unrestricted fetch*, consumes a resource occurring in a specific position of the unrestricted bag U via a linear head substitution of an unrestricted variable occurring in the head of the term. In this case, reduction results in an explicit substitution with U kept unaltered. Note that we check for the size of the linear bag C: in the case $\#(x, M) \neq \mathsf{size}(C)$, the term evolves to a linear failure via Rule $[\mathtt{R} : \mathtt{fail}^{\ell}]$ (see Example 10). This is another design choice: linear failure is prioritised in $u\lambda^{\lightning}_{\oplus}$.

Four rules show reduction to failure terms, and accumulate free variables involved in failed reductions. Rules $[\mathtt{R} : \mathtt{Fail}^{\ell}]$ and $[\mathtt{R} : \mathtt{Fail}^{!}]$ formalise the failure to evaluate an explicit substitution $M\langle\langle C \star U/x\rangle\rangle$. The former rule targets a linear failure, which occurs when the size of C does not match the number of occurrences of x. The multiset $\widetilde{y}$ preserves all free linear variables in M and C. The latter rule targets an *unrestricted failure*, which occurs when the head of the term is $x[i]$ and U_i (i.e., the i-th element of U) is empty. In this case, failure preserves the free linear variables in M and C excluding the head unrestricted occurrence $x[i]$ which is replaced by $\mathtt{fail}^{\emptyset}$.

Rules $[\mathtt{R} : \mathtt{Cons}_1]$ and $[\mathtt{R} : \mathtt{Cons}_2]$ describe reductions that lazily consume the failure term, when a term has $\mathtt{fail}^{\widetilde{x}}$ at its head position. The former rule consumes bags attached to it whilst preserving all its free linear variables; the latter rule consumes explicit substitution attached to it whilst also preserving all its free linear variables. The side condition $\#(z, \widetilde{x}) = \mathsf{size}(C)$ is necessary in Rule $[\mathtt{R} : \mathtt{Cons}_2]$ to avoid a clash with the premise of Rule $[\mathtt{R} : \mathtt{Fail}^{\ell}]$. Finally, Rules $[\mathtt{R} : \mathtt{ECont}]$ and $[\mathtt{R} : \mathtt{TCont}]$ state closure by the C and D contexts (cf. Def. 8).

Notice that the left-hand sides of the reduction rules in $u\lambda^{\lightning}_{\oplus}$ do not interfere with each other. As a result, reduction in $u\lambda^{\lightning}_{\oplus}$ satisfies a *diamond property*: for all $\mathbb{M} \in u\lambda^{\lightning}_{\oplus}$, if there exist $\mathbb{M}_1, \mathbb{M}_2 \in u\lambda^{\lightning}_{\oplus}$ such that $\mathbb{M} \longrightarrow \mathbb{M}_1$ and $\mathbb{M} \longrightarrow \mathbb{M}_2$, then there exists $\mathbb{N} \in u\lambda^{\lightning}_{\oplus}$ such that $\mathbb{M}_1 \longrightarrow \mathbb{N} \longleftarrow \mathbb{M}_2$ (see App. A).

▶ **Notation 2.** *As usual,* $\longrightarrow^*$ *denotes the reflexive-transitive closure of* $\longrightarrow$. *We write* $\mathbb{N} \longrightarrow_{[\mathtt{R}]} \mathbb{M}$ *to denote that* $[\mathtt{R}]$ *is the last (non-contextual) rule used in the step from* $\mathbb{N}$ *to* $\mathbb{M}$.

▶ **Example 9** (Cont. Example 3). We illustrate different reductions for $\lambda x.x$ and $\lambda x.x[i]$.

1. $M_1 = (\lambda x.x)(\wr N \int \star U)$ concerns a linear variable x with an linear bag containing one element. This is similar to the usual meaning of applying an identity function to a term: $(\lambda x.x)(\wr N \int \star U) \longrightarrow_{[\mathtt{R:Beta}]} x\langle\langle \wr N \int \star U / x \rangle\rangle \longrightarrow_{[\mathtt{R:Fetch}^\ell]} x\{|N/x|\}\langle\langle \mathtt{1} \star U / x \rangle\rangle = N \langle\langle \mathtt{1} \star U / x \rangle\rangle$, with a "garbage collector" that collects unused unrestricted resources.
2. $M_2 = (\lambda x.x)(\wr N_1 \int \cdot \wr N_2 \int \star U)$ concerns the case in which a linear variable x has a single occurrence but the linear bag has size two. Term M_2 reduces to a sum of failure terms: $(\lambda x.x)(\wr N_1 \int \cdot \wr N_2 \int \star U) \longrightarrow_{[\mathtt{R:Beta}]} x\langle\langle \wr N_1 \int \cdot \wr N_2 \int \star U / x \rangle\rangle \longrightarrow_{[\mathtt{R:Fail}^\ell]} \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}}$ for $C = \wr N_1 \int \cdot \wr N_2 \int$ and $\widetilde{y} = \mathsf{mlfv}(C)$.
3. $M_3 = (\lambda x.x[1])(\wr N \int \star \mathtt{1}^!)$ represents an abstraction of an unrestricted variable, which aims to consume the first element of the unrestricted bag. Because this bag is empty, M_3 reduces to failure:
$(\lambda x.x[1])(\wr N \int \star \mathtt{1}^!) \longrightarrow_{[\mathtt{R:Beta}]} x[1]\langle\langle \wr N \int \star \mathtt{1}^! / x \rangle\rangle \longrightarrow_{[\mathtt{R:fail}^\ell]} \mathtt{fail}^{\widetilde{y}}$,
for $\widetilde{y} = \mathsf{mlfv}(N)$. Notice that $0 = \#(x, x[1]) \neq \mathsf{size}(\wr N \int) = 1$, since there are no linear occurrences of x in $x[1]$.

▶ **Example 10.** To illustrate the need to check '$\mathsf{size}(C)$' in $[\mathtt{R:Fail}^!]$, consider the term $x[1]\langle\langle \wr M \int \star \mathtt{1}^! / x \rangle\rangle$, which features both a mismatch of linear bags for the linear variables to be substituted and an empty unrestricted bag with the need for the first element to be substituted. We check the size of the linear bag because we wish to prioritise the reduction of Rule $[\mathtt{R:Fail}^\ell]$. Hence, in case of a mismatch of linear resources we wish not to perform a reduction via Rule $[\mathtt{R:Fail}^!]$. This is a design choice: our semantics collapses linear failure at the earliest moment it arises.

▶ **Example 11** (Cont. Example 4). Self-applications of Δ_1 do not behave as an expected variation of a lazy reduction from Ω. Both $\Delta_1(\wr \Delta_1 \int \star \mathtt{1}^!)$ and $\Delta_1(\mathtt{1} \star \wr \Delta_1 \int^!)$ reduce to failure since the number of linear occurrences of x does not match the number of resources in the linear bag: $\Delta_1(\wr \Delta_1 \int \star \mathtt{1}^!) \longrightarrow (x(\wr x \int \star \mathtt{1}))\langle\langle \wr \Delta_1 \int \star \mathtt{1}^! / x \rangle\rangle \longrightarrow \mathtt{fail}^{\emptyset}$.

The term $\Delta_4(\mathtt{1} \star \wr \Delta_4 \int^!)$ also fails: the linear bag is empty and there is one linear occurrence of x in Δ_4. Note that $\Delta_4(\wr \Delta_4 \int \star \wr \Delta_4 \int^!)$ reduces to another application of Δ_4 before failing:

$$
\begin{aligned}
\Delta_4(\wr \Delta_4 \int \star \wr \Delta_4 \int^!) &= (\lambda x.(x[1](\wr x \int \star \mathtt{1}^!)))(\wr \Delta_4 \int \star \wr \Delta_4 \int^!) \\
&\longrightarrow_{[\mathtt{R:Beta}]} (x[1](\wr x \int \star \mathtt{1}^!))\langle\langle \wr \Delta_4 \int \star \wr \Delta_4 \int^! / x \rangle\rangle \\
&\longrightarrow_{[\mathtt{R:Fetch}^!]} (\Delta_4(\wr x \int \star \mathtt{1}^!))\langle\langle \wr \Delta_4 \int \star \wr \Delta_4 \int^! / x \rangle\rangle \\
&\longrightarrow^* \mathtt{fail}^{\emptyset}\langle\langle \wr x \int \star \mathtt{1}^! / y \rangle\rangle\langle\langle \wr \Delta_4 \int \star \wr \Delta_4 \int^! / x \rangle\rangle
\end{aligned}
$$

Differently from [17], there are terms in $u\lambda^{\lightning}_{\oplus}$ that when applied to each other behave similarly to Ω, namely $\Omega_{5,6}$, $\Omega_{6,5}$, and Ω_7 (Example 4).

▶ **Example 12** (Cont. Example 4). The following reductions illustrate different behaviours provided that subtle changes are made within $u\lambda^{\lightning}_{\oplus}$-terms:

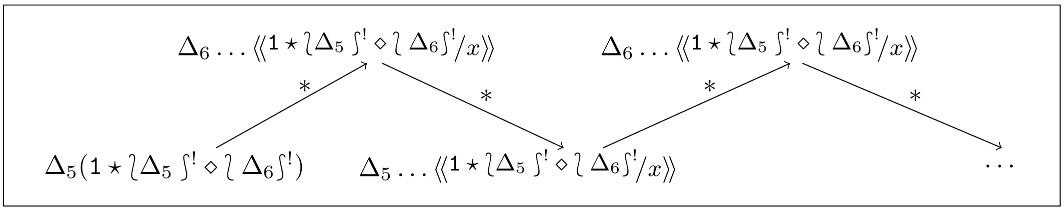

Figure 2 An Ω-like behaviour in $u\lambda^{\lightning}_{\oplus}$ (cf. Example 12).

- An interesting behaviour of $u\lambda^{\lightning}_{\oplus}$ is that variations of Δ can be applied to each other and appear alternately (highlighted in blue) in the functional position throughout the computation—this behaviour is illustrated in Fig. 2:
$\Omega_{5,6} = \Delta_5(1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^!)$
$= (\lambda x.(x[2](1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!)))(1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^!)$
$\longrightarrow_{[\texttt{R:Beta}]} (x[2](1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!))\langle\!\langle 1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (\Delta_6(1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!))\langle\!\langle 1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Beta}]} (y[1](1 \star \wr y[1] \int^! \diamond \wr y[2] \int^!)\langle\!\langle (1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!) / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (x[1](1 \star \wr y[1] \int^! \diamond \wr y[2] \int^!)\langle\!\langle (1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!) / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (\Delta_5(1 \star \wr y[1] \int^! \diamond \wr y[2] \int^!)\langle\!\langle (1 \star \wr x[1] \int^! \diamond \wr x[2] \int^!) / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_5 \int^! \diamond \wr \Delta_6 \int^! / x \rangle\!\rangle$
$\longrightarrow \ldots$
- Applications of Δ_7 into two unrestricted copies of Δ_7 behave as Ω producing a non-terminating behaviour. Letting $B = 1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!$, we have:
$\Omega_7 = (\lambda x.(x[2](1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!)))(1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^!)$
$\longrightarrow_{[\texttt{R:Beta}]} (x[2](1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!))\langle\!\langle 1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (\Delta_7(1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!))\langle\!\langle 1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Beta}]} (y[2](1 \star \wr y[1] \int^! \diamond \wr y[1] \int^!))\langle\!\langle B / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (x[1](1 \star \wr y[1] \int^! \diamond \wr y[1] \int^!))\langle\!\langle B / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^! / x \rangle\!\rangle$
$\longrightarrow_{[\texttt{R:Fetch}^!]} (\Delta_7(1 \star \wr y[1] \int^! \diamond \wr y[1] \int^!))\langle\!\langle B / y \rangle\!\rangle \langle\!\langle 1 \star \wr \Delta_7 \int^! \diamond \wr \Delta_7 \int^! / x \rangle\!\rangle$
$\longrightarrow \ldots$
Later on we will show that this term is well-formed (see Example 16) with respect to the intersection type system introduced in § 3.

3 Well-Formed Expressions via Intersection Types

We define *well-formed* $u\lambda^{\lightning}_{\oplus}$-expressions by relying on a non-idempotent intersection type system, based on the system by Bucciarelli et al. [4]. Our system for well-formed expressions subsumes the one in [17]: it uses *strict* and *multiset* types to check linear bags; moreover, it uses *list* and *tuple* types to check unrestricted bags. As in [17], we write "well-formedness" (of terms, bags, and expressions) to stress that, unlike usual type systems, our system can account for terms that may reduce to the failure term (cf. Remark 18).

▶ **Definition 13** (Types for $u\lambda^{\lightning}_{\oplus}$). *We define* strict, multiset, list, *and* tuple types.

(Strict)	σ, τ, δ	$::=$	$\mathbf{unit} \mid (\pi, \eta) \to \sigma$	*(List)*	η, ϵ	$::=$	$\sigma \mid \epsilon \diamond \eta$
(Multiset)	π, ζ	$::=$	$\bigwedge_{i \in I} \sigma_i \mid \omega$	*(Tuple)*	(π, η)		

A strict type can be the **unit** type or a functional type $(\pi, \eta) \rightarrow \sigma$, where (π, η) is a tuple type and σ is a strict type. Multiset types can be either the empty type ω or an intersection of strict types $\bigwedge_{i \in I} \sigma_i$, with I non-empty. The operator $\wedge$ is commutative, associative, non-idempotent, that is, $\sigma \wedge \sigma \neq \sigma$, with identity ω. The intersection type $\bigwedge_{i \in I} \sigma_i$ is the type of a linear bag; the cardinality of I corresponds to its size.

A list type can be either an strict type σ or the composition $\epsilon \diamond \eta$ of two list types ϵ and η. We use the list type $\epsilon \diamond \eta$ to type the concatenation of two unrestricted bags. A tuple type (π, η) types the concatenation of a linear bag of type π with an unrestricted bag of type η. Notice that a list type $\epsilon \diamond \eta$ can be recursively unfolded into a finite composition of strict types $\sigma_1 \diamond \ldots \diamond \sigma_n$, for some $n \geq 1$. In this case the length of $\epsilon \diamond \eta$ is n and that σ_i is its i-th strict type, for $1 \leq i \leq n$.

▶ **Notation 3.** *Given $k \geq 0$, we write σ^k to stand for $\sigma \wedge \cdots \wedge \sigma$ (k times, if $k > 0$) or for ω (if $k = 0$). Similarly, $\hat{x} : \sigma^k$ stands for $x : \sigma, \cdots, x : \sigma$ (k times, if $k > 0$) or for $x : \omega$ (if $k = 0$). Given $k \geq 1$, we write $x^! : \eta$ to stand for $x[1] : \eta_1, \cdots, x[k] : \eta_k$.*

▶ **Notation 4** ($\eta \propto \epsilon$)**.** *Let ϵ and η be two list types, with the length of ϵ greater or equal to that of η. Let us write ϵ_i and η_i to denote the i-th strict type in ϵ and η, respectively. We write $\eta \propto \epsilon$ meaning the initial sublist, whenever there exist ϵ' and ϵ'' such that: i) $\epsilon = \epsilon' \diamond \epsilon''$; ii) the size of ϵ' is that of η; iii) for all i, $\epsilon'_i = \eta_i$.*

Linear contexts range over $\Gamma, \Delta, \ldots$ and *unrestricted contexts* range over $\Theta, \Upsilon, \ldots$. They are defined by the following grammar:

$$\Gamma, \Delta ::= \text{-} \mid x : \sigma \mid \Gamma, x : \sigma \qquad \Theta, \Upsilon ::= \text{-} \mid x^! : \eta \mid \Theta, x^! : \eta$$

The empty linear/unrestricted type assignment is denoted '-'. Linear variables can occur more than once in a linear context; they are assigned only strict types. For instance, $x : (\tau, \sigma) \rightarrow \tau, x : \tau$ is a valid context: it means that x can be of both type $(\tau, \sigma) \rightarrow \tau$ and τ. In contrast, unrestricted variables can occur at most once in unrestricted contexts; they are assigned only list types. The multiset of linear variables in Γ is denoted as $\texttt{dom}(\Gamma)$; similarly, $\texttt{dom}(\Theta)$ denotes the set of unrestricted variables in Θ.

Judgements are of the form $\Theta; \Gamma \models \mathbb{M} : \sigma$, where the left-hand side contexts are separated by ";" and $\mathbb{M} : \sigma$ means that $\mathbb{M}$ has type σ. We write $\models \mathbb{M} : \sigma$ to denote $\text{-}\,;\,\text{-} \models \mathbb{M} : \sigma$.

▶ **Definition 14** (Well-formed $u\lambda^{\lightning}_{\oplus}$ expressions)**.** *An expression $\mathbb{M}$ is well-formed (wf, for short) if there exist Γ, Θ and τ such that $\Theta; \Gamma \models \mathbb{M} : \tau$ is entailed via the rules in Fig. 3.*

We describe the well-formedness rules in Fig. 3.

- Rules $[\texttt{F} : \texttt{var}^{\ell}]$ and $[\texttt{F} : \texttt{var}^!]$ assign types to linear and unrestricted variables, respectively.
- Rule $[\texttt{F} : \texttt{var}^!]$ resembles the *copy* rule [6] where we use a linear copy of an unrestricted variable $x[i]$ of type σ, typed with $x^! : \eta$, and type the linear copy with the corresponding strict type η_i which in this case the linear copy x would have type equal to σ.
- Rules $[\texttt{F} : \texttt{1}^{\ell}]$ and $[\texttt{F} : \texttt{1}^!]$ assign types to the empty linear/unrestricted bag: $\texttt{1}$ has type ω, whereas $\texttt{1}^!$ has an arbitrary strict type σ. Arbitrariness is allowed since the substitution of an unrestricted variable for $1^!$ leads to a $\texttt{fail}$ term (Rule $[\texttt{R} : \texttt{Fail}^!]$), which has an arbitrary strict type.
- Rule $[\texttt{F} : \texttt{abs}]$ assigns type $(\sigma^k, \eta) \rightarrow \tau$ to an abstraction $\lambda z.M$, provided that the unrestricted occurrences of z may be typed by the unrestricted context containing $z^! : \eta$, the linear occurrences of z are typed with the linear context containing $\hat{z} : \sigma^k$, for some $k \geq 0$, and there are no other linear occurrences of z in the linear context Γ.

$$[\texttt{F:var}^{\ell}]\ \frac{}{\Theta; x:\sigma \models x:\sigma} \qquad [\texttt{F:var}^{!}]\ \frac{\Theta, x^{!}:\eta; x:\eta_i, \Delta \models x:\sigma}{\Theta, x^{!}:\eta; \Delta \models x[i]:\sigma} \qquad [\texttt{F:1}^{\ell}]\ \frac{}{\Theta; \text{-} \models 1:\omega}$$

$$[\texttt{F:1}^{!}]\ \frac{}{\Theta; \text{-} \models 1^{!}:\sigma} \qquad [\texttt{F:abs}]\ \frac{\Theta, z^{!}:\eta; \Gamma, \hat{z}:\sigma^{k} \models M:\tau \quad z \notin \texttt{dom}(\Gamma)}{\Theta; \Gamma \models \lambda z.M : (\sigma^{k}, \eta) \to \tau}$$

$$[\texttt{F:app}]\ \frac{\Theta; \Gamma \models M : (\sigma^{j}, \eta) \to \tau \qquad \Theta; \Delta \models B : (\sigma^{k}, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma, \Delta \models M\ B : \tau} \qquad [\texttt{F:ex-sub}]\ \frac{\Theta, x^{!}:\eta; \Gamma, \hat{x}:\sigma^{j} \models M:\tau \qquad \Theta; \Delta \models B : (\sigma^{k}, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma, \Delta \models M\langle\langle B/x \rangle\rangle : \tau}$$

$$[\texttt{F:bag}]\ \frac{\Theta; \Gamma \models C : \sigma^{k} \qquad \Theta; \text{-} \models U : \eta}{\Theta; \Gamma \models C \star U : (\sigma^{k}, \eta)} \qquad [\texttt{F:bag}^{\ell}]\ \frac{\Theta; \Gamma \models M : \sigma \qquad \Theta; \Delta \models C : \sigma^{k}}{\Theta; \Gamma, \Delta \models \wr M \wr \cdot C : \sigma^{k+1}}$$

$$[\texttt{F:bag}^{!}]\ \frac{\Theta; \text{-} \models M : \sigma}{\Theta; \text{-} \models \wr M \wr^{!} : \sigma} \qquad [\texttt{F:}\diamond\,\texttt{bag}^{!}]\ \frac{\Theta; \text{-} \models U : \epsilon \qquad \Theta; \text{-} \models V : \eta}{\Theta; \text{-} \models U \diamond V : \epsilon \diamond \eta} \qquad [\texttt{F:fail}]\ \frac{\texttt{dom}(\Gamma) = \widetilde{x}}{\Theta; \Gamma \models \texttt{fail}^{\widetilde{x}} : \tau}$$

$$[\texttt{F:sum}]\ \frac{\Theta; \Gamma \models \mathbb{M} : \sigma \qquad \Theta; \Gamma \models \mathbb{N} : \sigma}{\Theta; \Gamma \models \mathbb{M} + \mathbb{N} : \sigma} \qquad [\texttt{F:weak}]\ \frac{\Theta; \Gamma \models M : \sigma \quad x \notin \texttt{dom}(\Gamma)}{\Theta; \Gamma, x:\omega \models M : \sigma}$$

Figure 3 Well-formedness rules for $u\lambda^{\lightning}_{\oplus}$ (cf. Def. 14). In Rules [F:app] and [F:ex-sub]: $k, j \geq 0$.

- Rules [F:app] and [F:ex-sub] (for application and explicit substitution, resp.) use the condition $\eta \propto \epsilon$ (cf. Notation 4), which captures the portion of the unrestricted bag that is effectively used in a term: it ensures that ϵ can be decomposed into some ϵ' and ϵ'', such that each type component ϵ'_i matches with η_i. If this requirement is satisfied, Rule [F:app] types an application $M\ B$ given that M has a functional type in which the left of the arrow is a tuple type (σ^j, η) whereas the bag B is typed with tuple (σ^k, ϵ). Similarly, Rule [F:ex-sub] types the term $M\langle\langle B/x \rangle\rangle$ provided that B has the tuple type (σ^k, ϵ) and M is typed with the variable x having linear type assignment σ^j and unrestricted type assignment η.

▶ Remark 15. Differently from intersection type systems [4, 16], in Rules [F:app] and [F:ex-sub] there is no equality requirement between j and k, as we would like to capture terms that fail due to a mismatch in resources: we only require that the linear part of the tuples are composed of the same strict type, say σ. As a term can take an unrestricted bag with arbitrary size we only require that the elements of the unrestricted bag that are used have a "consistent" type, i.e., the type of the unrestricted bag satisfies the relation $\propto$ with the unrestricted fragment of the corresponding tuple type.

There are four rules for bags:

- Rule [F : bag$^!$] types an unrestricted bag $\wr M \wr^!$ with the type σ of M. Note that $\wr x \wr^!$, an unrestricted bag containing a linear variable x, is not well-formed, whereas $\wr x[i] \wr^!$ is well-formed.
- Rule [F : bag] assigns the tuple type (σ^k, η) to the concatenation of a linear bag of type σ^k with an unrestricted bag of type η.
- Rules [F : bag$^\ell$] and [F: $\diamond$ bag$^!$] type the concatenation of linear and unrestricted bags.
- Rule [F:1$^!$] allows an empty unrestricted bag to have an arbitrary σ type since it may be referred to by a variable for substitution: we must be able to compare its type with the type of unrestricted variables that may consume the empty bag (this reduction would inevitably lead to failure).

As in [17], Rule [F:fail] handles the failure term, and is the main difference with respect to standard type systems. Rules for sums and weakening ([F : sum] and [F : weak]) are standard.

▶ **Example 16** (Cont. Example 12). Term $\Delta_7 := \lambda x.x[2](1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!)$ is well-formed, as ensured by the judgement $\Theta; \text{-} \models \Delta_7 : (\omega, \sigma' \diamond (\sigma^j, \sigma' \diamond \sigma') \to \tau) \to \tau$, whose derivation is given below:

- Π_3 is the derivation of $\Theta, x^! : \eta; \text{-} \models \wr x[1] \int^! : \sigma'$, for $\eta = \sigma' \diamond (\sigma^j, \sigma' \diamond \sigma') \to \tau$.
- Π_4 is the derivation: $\Theta, x^! : \eta; \text{-} \models x[2] : (\sigma^j, \sigma' \diamond \sigma') \to \tau$
- Π_5 is the derivation: $\Theta, x^! : \eta; x : \omega \models (1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!) : (\omega, \sigma' \diamond \sigma')$

Therefore,

$$\dfrac{\dfrac{\Pi_5 \quad \Pi_4 \quad \sigma' \diamond \sigma' \propto \sigma' \diamond \sigma'}{\Theta, x^! : \eta; x : \omega \models x[2](1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!) : \tau}\;[\texttt{F:app}]}{\Theta; \text{-} \models \underbrace{\lambda x.(x[2](1 \star \wr x[1] \int^! \diamond \wr x[1] \int^!))}_{\Delta_7} : (\omega, \eta) \to \tau}\;[\texttt{F:abs}]$$

Well-formed expressions satisfy subject reduction (SR); see App. B for a proof.

▶ **Theorem 17** (SR in $u\lambda^{\lightning}_{\oplus}$). *If* $\Theta; \Gamma \models \mathbb{M} : \tau$ *and* $\mathbb{M} \longrightarrow \mathbb{M}'$ *then* $\Theta; \Gamma \models \mathbb{M}' : \tau$.

Proof. By structural induction on the reduction rules. We proceed by analysing the rule applied in $\mathbb{M}$. An interesting case occurs when the rule is $[\texttt{F : Fetch}^!]$: Then $\mathbb{M} = M \langle\langle C \star U / x \rangle\rangle$, where $U = \wr N_1 \int^! \diamond \cdots \diamond \wr N_l \int^!$ and $\mathsf{head}(M) = x[i]$. The reduction is as follows:

$$[\texttt{R : Fetch}^!]\; \dfrac{\mathsf{head}(M) = x[i] \quad U_i = \wr N_i \int^!}{M \langle\langle C \star U / x \rangle\rangle \longrightarrow M\{\!| N_i / x[i] |\!\}\langle\langle C \star U / x \rangle\rangle}$$

By hypothesis, one has the derivation:

$$[\texttt{F:ex-sub}]\; \dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau \qquad [\texttt{F:bag}]\; \dfrac{\overset{\Pi}{\Theta; \cdot \models U : \epsilon} \quad \Theta; \Delta \models C : \sigma^k}{\Theta; \Delta \models C \star U : (\sigma^k, \epsilon)} \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M \langle\langle C \star U / x \rangle\rangle : \tau}$$

where Π has the form

$$[\texttt{F:}\diamond\,\texttt{bag}^!]\; \dfrac{[\texttt{F:bag}^!]\; \dfrac{\Theta; \cdot \models N_1 : \epsilon_1}{\Theta; \cdot \models \wr N_1 \int^! : \epsilon_1} \quad \cdots \quad [\texttt{F:bag}^!]\; \dfrac{\Theta; \cdot \models N_l : \epsilon_l}{\Theta; \cdot \models \wr N_l \int^! : \epsilon_l}}{\Theta; \cdot \models \wr N_1 \int^! \diamond \cdots \diamond \wr N_l \int^! : \epsilon}$$

with $\Gamma = \Gamma', \Delta$. Notice that if $\epsilon_i = \delta$ and $\eta \propto \epsilon$ then $\eta_i = \delta$. By Lemma 35, there exists a derivation Π_1 of $\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!| N_i / x[i] |\!\} : \tau$. Therefore, we have:

$$[\texttt{F:ex-sub}]\; \dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!| N_1 / x[i] |\!\} : \tau \qquad [\texttt{F:bag}]\; \dfrac{\Theta; \cdot \models U : \epsilon \quad \Theta; \Delta \models C : \sigma^k}{\Theta; \Delta \models C \star U : (\sigma^k, \epsilon)} \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\{\!| N_i / x[i] |\!\}\langle\langle C \star U / x \rangle\rangle : \tau}$$

◀

▶ Remark 18 (Well-Formed vs Well-Typed Expressions). Our type system (and Theorem 17) can be specialised to the case of *well-typed* expressions that do not contain (and never reduce to) the failure term. In particular, Rules [F:app] and [F:ex-sub] would need to check that $\sigma^k = \sigma^j$, as failure can be caused due to a mismatch of linear resources. A difference between well typed and well formed expressions is that the former satisfy subject expansion, but the latter do not: expressions that lead to failure can be ill-typed yet failure itself is well-formed.

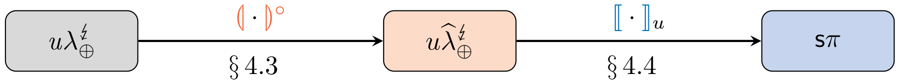

Figure 4 Our two-step approach to encode $u\lambda^{\lightning}_{\oplus}$ into sπ.

4 A Typed Encoding of $u\lambda^{\lightning}_{\oplus}$ into Concurrent Processes

We encode $u\lambda^{\lightning}_{\oplus}$ into sπ, a session π-calculus that stands on a Curry-Howard correspondence between linear logic and session types (§ 4.1). We extend the two-step approach that we devised in [17] for the sub-calculus $\lambda^{\lightning}_{\oplus}$ (with linear resources only) (cf. Fig. 4). First, in § 4.3, we define an encoding $(\!|\cdot|\!)^{\circ}$ from well-formed expressions in $u\lambda^{\lightning}_{\oplus}$ to well-formed expressions in a variant of $u\lambda^{\lightning}_{\oplus}$ with *sharing*, dubbed $u\widehat{\lambda}^{\lightning}_{\oplus}$ (§ 4.2). Then, in § 4.4, we define an encoding $[\![\cdot]\!]_u$ (for a name u) from well-formed expressions in $u\widehat{\lambda}^{\lightning}_{\oplus}$ to well-typed processes in sπ.

We prove that $(\!|\cdot|\!)^{\circ}$ and $[\![\cdot]\!]_u$ satisfy well-established correctness criteria [9, 14]: *type preservation*, *operational completeness*, *operational soundness*, and *success sensitiveness* (cf. App. E.1). Because $u\lambda^{\lightning}_{\oplus}$ includes unrestricted resources, the results given here strictly generalise those in [17].

4.1 sπ: A Session-Typed π-Calculus

sπ is a π-calculus with *session types* [11, 12], which ensure that the endpoints of a channel perform matching actions. We consider the full process framework in [5], including constructs for specifying labelled choices and client/server connections; they will be useful to codify unrestricted resources and variables in $u\lambda^{\lightning}_{\oplus}$. Following [6, 18], s$\pi$ stands on a Curry-Howard correspondence between session types and a linear logic with dual modalities/types ($\&A$ and $\oplus A$), which define *non-deterministic* session behaviour. As in [6, 18], in sπ, cut elimination corresponds to communication, proofs to processes, and propositions to session types.

Syntax. *Names* $x, y, z, w \ldots$ denote the endpoints of protocols specified by session types. We write $P\{x/y\}$ for the capture-avoiding substitution of x for y in process P.

▶ **Definition 19** (Processes). *The syntax of* sπ *processes is given by the grammar below.*

$$\begin{aligned} P, Q ::= \;& \mathbf{0} \mid \overline{x}(y).P \mid x(y).P \mid x.\mathtt{l}_i; P \mid x.\mathtt{case}_{i\in I}\{\mathtt{l}_i : P_i\} \mid x.\overline{\mathtt{close}} \mid x.\mathtt{close}; P \\ & \mid (P \mid Q) \mid [x \leftrightarrow y] \mid (\nu x)P \mid \; !x(y).P \mid \overline{x}?(y).P \\ & \mid x.\overline{\mathtt{some}}; P \mid x.\overline{\mathtt{none}} \mid x.\mathtt{some}_{(w_1,\cdots,w_n)}; P \mid (P \oplus Q) \end{aligned}$$

Process $\mathbf{0}$ denotes inaction. Process $\overline{x}(y).P$ sends a fresh name y along x and then continues as P. Process $x(y).P$ receives a name z along x and then continues as $P\{z/y\}$. Process $x.\mathtt{case}_{i\in I}\{\mathtt{l}_i : P_i\}$ is a branching construct, with labelled alternatives indexed by the finite set I: it awaits a choice on x with continuation P_j for each $j \in I$. Process $x.\mathtt{l}_i; P$ selects on x the alternative indexed by i before continuing as P. Processes $x.\overline{\mathtt{close}}$ and $x.\mathtt{close}; P$ are complementary actions for closing session x. We sometimes use the shorthand notations $\overline{y}[\,]$ and $y[\,]; P$ to stand for $y.\overline{\mathtt{close}}$ and $y.\mathtt{close}; P$, respectively. Process $P \mid Q$ is the parallel execution of P and Q. The forwarder process $[x \leftrightarrow y]$ denotes a bi-directional link between sessions x and y. Process $(\nu x)P$ denotes the process P in which name x is kept private (local) to P. Process $!x(y).P$ defines a server that spawns copies of P upon requests on x. Process $\overline{x}?(y).P$ denotes a client that connects to a server by sending the fresh name y on x.

$$\begin{array}{ll}
\overline{x}(y).Q \mid x(y).P \longrightarrow (\nu y)(Q \mid P) & x.\overline{\mathtt{some}}; P \mid x.\mathtt{some}_{(w_1,\cdots,w_n)}; Q \longrightarrow P \mid Q \\
Q \longrightarrow Q' \Rightarrow P \oplus Q \longrightarrow P \oplus Q' & x.\overline{\mathtt{close}} \mid x.\mathtt{close}; P \longrightarrow P \\
x.\mathtt{l}_i; Q \mid x.\mathtt{case}_{i\in I}\{\mathtt{l}_i : P_i\} \longrightarrow Q \mid P_i & !x(y).Q \mid \overline{x}?(y).P \longrightarrow (\nu x)(!x(y).Q \mid (\nu y)(Q \mid P)) \\
(\nu x)([x \leftrightarrow y] \mid P) \longrightarrow P\{y/x\} \quad (x \neq y) & P \equiv P' \wedge P' \longrightarrow Q' \wedge Q' \equiv Q \Rightarrow P \longrightarrow Q \\
Q \longrightarrow Q' \Rightarrow P \mid Q \longrightarrow P \mid Q' & P \longrightarrow Q \Rightarrow (\nu y)P \longrightarrow (\nu y)Q \\
x.\overline{\mathtt{none}} \mid x.\mathtt{some}_{(w_1,\cdots,w_n)}; Q \longrightarrow w_1.\overline{\mathtt{none}} \mid \cdots \mid w_n.\overline{\mathtt{none}} &
\end{array}$$

Figure 5 Reduction for $\mathsf{s}\pi$

The remaining constructs come from [5] and introduce non-deterministic sessions which *may* provide a session protocol *or* fail. Process $x.\overline{\mathtt{some}}; P$ confirms that the session on x will execute and continues as P. Process $x.\overline{\mathtt{none}}$ signals the failure of implementing the session on x. Process $x.\mathtt{some}_{(w_1,\cdots,w_n)}; P$ specifies a dependency on a non-deterministic session x. This process can either (i) synchronise with an action $x.\overline{\mathtt{some}}$ and continue as P, or (ii) synchronise with an action $x.\overline{\mathtt{none}}$, discard P, and propagate the failure on x to $(w_1, \cdots, w_n)$, which are sessions implemented in P. When x is the only session implemented in P, there is no tuple of dependencies $(w_1, \cdots, w_n)$ and so we write simply $x.\mathtt{some}; P$. Finally, process $P \oplus Q$ denotes a non-deterministic choice between P and Q. We shall often write $\bigoplus_{i\in\{1,\cdots,n\}} P_i$ to stand for $P_1 \oplus \cdots \oplus P_n$. In $(\nu y)P$ and $x(y).P$ the occurrence of name y is binding, with scope P. The set of free names of P is denoted by $\mathit{fn}(P)$.

Semantics. The *reduction relation* of $\mathsf{s}\pi$ specifies the computations that a process performs on its own (cf. Fig. 5). It is closed by *structural congruence*, denoted $\equiv$, which expresses basic identities for processes and the non-collapsing nature of non-determinism (cf. App. C).

The first reduction rule formalises communication, which concerns bound names only (internal mobility), as y is bound in $\overline{x}(y).Q$ and $x(y).P$. Reduction for the forwarder process leads to a substitution. The reduction rule for closing a session is self-explanatory, as is the rule in which prefix $x.\overline{\mathtt{some}}$ confirms the availability of a non-deterministic session. When a non-deterministic session is not available, $x.\overline{\mathtt{none}}$ triggers this failure to all dependent sessions $w_1, \ldots, w_n$; this may in turn trigger further failures (i.e., on sessions that depend on $w_1, \ldots, w_n$). The remaining rules define contextual reduction with respect to restriction, composition, and non-deterministic choice.

Type System Session types govern the behaviour of the names of a process. An assignment $x : A$ enforces the use of name x according to the protocol specified by A.

▶ **Definition 20** (Session Types). *Session types are given by*

$$A, B ::= \quad \bot \mid \mathbf{1} \mid A \otimes B \mid A \parr B \mid \ \oplus_{i\in I} \{\mathtt{l}_i : A_i\} \mid \&_{i\in I}\{\mathtt{l}_i : A_i\} \mid {!A} \mid {?A} \mid \&A \mid \ \oplus A$$

The multiplicative units $\bot$ and $\mathbf{1}$ are used to type closed session endpoints. We use $A \otimes B$ to type a name that first outputs a name of type A before proceeding as specified by B. Similarly, $A \parr B$ types a name that first inputs a name of type A before proceeding as specified by B. Then, $!A$ types a name that repeatedly provides a service specified by A. Dually, $?A$ is the type of a name that can connect to a server offering A. Types $\oplus_{i\in I}\{\mathtt{l}_i : A_i\}$ and $\&_{i\in I}\{\mathtt{l}_i : A_i\}$ are assigned to names that can select and offer a labelled choice, respectively. Then we have the two modalities introduced in [5]. We use $\&A$ as the type of a (non-deterministic) session that *may produce* a behaviour of type A. Dually, $\oplus A$ denotes the type of a session that *may consume* a behaviour of type A.

$$[\mathtt{Tid}]\ \frac{}{[x \leftrightarrow y] \vdash x{:}A, y{:}\overline{A};\ \Theta} \qquad [\mathtt{T1}]\ \frac{}{x.\overline{\mathtt{close}} \vdash x : \mathbf{1};\ \Theta} \qquad [\mathtt{T}\bot]\ \frac{P \vdash \Delta; \Theta}{x.\mathtt{close}; P \vdash x{:}\bot, \Delta;\ \Theta}$$

$$[\mathtt{T}\otimes]\ \frac{P \vdash \Delta, y : A;\ \Theta \quad Q \vdash \Delta', x : B;\ \Theta}{\overline{x}(y).(P \mid Q) \vdash \Delta, \Delta', x : A \otimes B;\ \Theta} \qquad [\mathtt{T}\parr]\ \frac{P \vdash \Delta, y : C, x : D;\ \Theta}{x(y).P \vdash \Delta, x : C \parr D;\ \Theta}$$

$$[\mathtt{T}\oplus^{\mathtt{x}}_{\widetilde{\mathtt{w}}}]\ \frac{P \vdash \widetilde{w} : \&\Delta, x : A;\ \Theta}{x.\mathtt{some}_{\widetilde{w}}; P \vdash \widetilde{w}{:}\&\Delta, x{:} \oplus A;\ \Theta} \qquad [\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}]\ \frac{P \vdash \Delta, x : A;\ \Theta}{x.\overline{\mathtt{some}}; P \vdash \Delta, x : \&A;\ \Theta}$$

$$[\mathtt{T}\&^{\mathtt{x}}]\ \frac{}{x.\overline{\mathtt{none}} \vdash x : \&A;\ \Theta} \qquad [\mathtt{T}\oplus]\ \frac{P \vdash \&\Delta;\ \Theta \qquad Q \vdash \&\Delta;\ \Theta}{P \oplus Q \vdash \&\Delta;\ \Theta}$$

$$[\mathtt{T}\oplus_{\mathtt{i}}]\ \frac{P \vdash \Delta, x : A_i; \Theta}{x.\mathtt{l}_i; P \vdash \Delta, x : \oplus_{i \in I}\{\mathtt{l}_i : A_i\}; \Theta} \qquad [\mathtt{T}\&]\ \frac{P_i \vdash \Delta, x : A_i; \Theta \quad (\forall i \in I)}{x.\mathtt{case}_{i \in I}\{\mathtt{l}_i : P_i\} \vdash \Delta, x : \&_{i \in I}\{\mathtt{l}_i : A_i\}; \Theta}$$

$$[\mathtt{T?}]\ \frac{P \vdash \Delta; x : A, \Theta}{P \vdash \Delta, x : ?A; \Theta} \qquad [\mathtt{T!}]\ \frac{P \vdash y : A; \Theta}{!x(y).P \vdash x : !A; \Theta} \qquad [\mathtt{Tcopy}]\ \frac{P \vdash \Delta, y : A; x : A, \Theta}{\overline{x}?(y).P \vdash \Delta; x : A, \Theta}$$

Figure 6 Typing rules for $\mathsf{s}\pi$.

The two endpoints of a session should be *dual* to ensure absence of communication errors. The dual of a type A is denoted $\overline{A}$. Duality corresponds to negation $(\cdot)^\perp$ in linear logic [5].

▶ **Definition 21** (Duality). *Duality on types is given by:*

$$\begin{array}{ccccc} \overline{\mathbf{1}} = \bot & \overline{\bot} = \mathbf{1} & \overline{A \otimes B} = \overline{A} \parr \overline{B} & \overline{\oplus_{i \in I}\{\mathtt{l}_i : A_i\}} = \&_{i \in I}\{\mathtt{l}_i : \overline{A_i}\} & \overline{\oplus A} = \&\overline{A} \\ \overline{!A} = ?\overline{A} & \overline{?A} = !\overline{A} & \overline{A \parr B} = \overline{A} \otimes \overline{B} & \overline{\&_{i \in I}\{\mathtt{l}_i : A_i\}} = \oplus_{i \in I}\{\mathtt{l}_i : \overline{A_i}\} & \overline{\&A} = \oplus\overline{A} \end{array}$$

Judgements are of the form $P \vdash \Delta; \Theta$, where P is a process, Δ is the linear context, and Θ is the unrestricted context. Both Δ and Θ contain assignments of types to names, but satisfy different substructural principles: while Θ satisfies weakening, contraction and exchange, Δ only satisfies exchange. The empty context is denoted '$\cdot$'. We write $\&\Delta$ to denote that all assignments in Δ have a non-deterministic type, i.e., $\Delta = w_1{:}\&A_1, \ldots, w_n{:}\&A_n$, for some $A_1, \ldots, A_n$. The typing judgement $P \vdash \Delta$ corresponds to the logical sequent for classical linear logic, which can be recovered by erasing processes and name assignments.

Typing rules for processes in Fig. 6 correspond to proof rules in linear logic; we discuss some of them. Rule [Tid] interprets the identity axiom using the forwarder process. Rules [T1] and [$\mathtt{T}\bot$] type the process constructs for session termination. Rules [$\mathtt{T}\otimes$] and [$\mathtt{T}\parr$] type output and input of a name along a session, resp. The last four rules are used to type process constructs related to non-determinism and failure. Rules [$\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}$] and [$\mathtt{T}\&^{\mathtt{x}}$] introduce a session of type $\&A$, which may produce a behaviour of type A: while the former rule covers the case in which $x : A$ is indeed available, the latter rule formalises the case in which $x : A$ is not available (i.e., a failure). Given a sequence of names $\widetilde{w} = w_1, \ldots, w_n$, Rule [$\mathtt{T}\oplus^{\mathtt{x}}_{\widetilde{\mathtt{w}}}$] accounts for the possibility of not being able to consume the session $x : A$ by considering sessions different from x as potentially not available. Rule [$\mathtt{T}\oplus$] expresses non-deterministic choice of processes P and Q that implement non-deterministic behaviours only. Finally, Rule [$\mathtt{T}\oplus_{\mathtt{i}}$] and [$\mathtt{T}\&$] correspond, resp., to selection and branching: the former provides a selection of behaviours along x as long as P is guarded with the i-th behaviour; the latter offers a labelled choice where each behaviour A_i is matched to a corresponding P_i.

The type system enjoys type preservation, a result that follows from the cut elimination property in linear logic; it ensures that the observable interface of a system is invariant under reduction. The type system also ensures other properties for well-typed processes (e.g. global progress, strong normalisation, and confluence); see [5] for details.

▶ **Theorem 22** (Type Preservation [5]). *If $P \vdash \Delta;\ \Theta$ and $P \longrightarrow Q$ then $Q \vdash \Delta;\ \Theta$.*

4.2 $u\widehat{\lambda}^{\lightning}_{\oplus}$: An Auxiliary Calculus With Sharing

To facilitate the encoding of $u\lambda^{\lightning}_{\oplus}$ into $\mathtt{s}\pi$, we define $u\widehat{\lambda}^{\lightning}_{\oplus}$: an auxiliary calculus whose constructs are inspired by the work of Gundersen et al. [10], Ghilezan et al. [8], and Kesner and Lengrand [13]. The syntax of $u\widehat{\lambda}^{\lightning}_{\oplus}$ only modifies the syntax of terms, which is defined by the grammar below; variables $x[*]$, bags B, and expressions $\mathbb{M}$ are as in Definition 1.

$$
\begin{aligned}
\text{(Terms)}\quad M,N,L ::= \;& x[*] \mid \lambda x.(M[\widetilde{x} \leftarrow x]) \mid (M\ B) \mid M\langle\!\lvert N/x \rvert\!\rangle \mid M\llbracket U/x \rrbracket \\
\mid\;& \mathtt{fail}^{\widetilde{x}} \mid M[\widetilde{x} \leftarrow x] \mid (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle
\end{aligned}
$$

We consider the *sharing construct* $M[\widetilde{x} \leftarrow x]$ and two kinds of explicit substitutions: the *explicit linear substitution*, written $M\langle\!\lvert N/x \rvert\!\rangle$, and the *explicit unrestricted substitution*, written $M\llbracket U/x \rrbracket$. The term $M[\widetilde{x} \leftarrow x]$ defines the sharing of variables $\widetilde{x}$ occurring in M using the linear variable x. We shall refer to x as *sharing variable* and to $\widetilde{x}$ as *shared variables.* A linear variable is only allowed to appear once in a term. Notice that $\widetilde{x}$ can be empty: $M[\leftarrow x]$ expresses that x does not share any variables in M. As in $u\lambda^{\lightning}_{\oplus}$, the term $\mathtt{fail}^{\widetilde{x}}$ explicitly accounts for failed attempts at substituting the variables in $\widetilde{x}$.

We summarise some requirements. In $M[\widetilde{x} \leftarrow x]$, we require: (i) every $x_i \in \widetilde{x}$ occurs exactly once in M and that (ii) x_i is not a sharing variable. The occurrence of x_i can appear within the fail term $\mathtt{fail}^{\widetilde{y}}$, if $x_i \in \widetilde{y}$. In the explicit linear substitution $M\langle\!\lvert N/x \rvert\!\rangle$, we require: the variable x has to occur in M; x cannot be a sharing variable; and x cannot be in an explicit linear substitution occurring in M; all free *linear* occurrences of x in M are bound. In the explicit unrestricted substitution $M\llbracket U/x \rrbracket$, we require: all free *unrestricted* occurrences of x in M are bound; x cannot be in an explicit unrestricted substitution occurring in M. This way, e.g., $M'\langle\!\lvert L/x \rvert\!\rangle\langle\!\lvert N/x \rvert\!\rangle$ and $M'\langle\!\lvert U'/x \rvert\!\rangle\langle\!\lvert U/x \rvert\!\rangle$ are not valid terms in $u\widehat{\lambda}^{\lightning}_{\oplus}$.

The following congruence will be important when proving encoding correctness.

▶ **Definition 23.** *The congruence $\equiv_\lambda$ for $u\widehat{\lambda}^{\lightning}_{\oplus}$ on terms and expressions is given by the identities below.*

$$
\begin{array}{rcl}
M\llbracket U/x \rrbracket & \equiv_\lambda & M,\ x \notin M \\
(MB)\langle\!\lvert N/x \rvert\!\rangle & \equiv_\lambda & (M\langle\!\lvert N/x \rvert\!\rangle)B,\ x \notin \mathsf{fv}(B) \\
(MB)\llbracket U/x \rrbracket & \equiv_\lambda & (M\llbracket U/x \rrbracket)B,\ x \notin \mathsf{fv}(B) \\
(MA)[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle & \equiv_\lambda & (M[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle)A,\ x_i \in \widetilde{x} \Rightarrow x_i \notin \mathsf{fv}(A) \\
M[\widetilde{y} \leftarrow y]\langle\!\langle A/y \rangle\!\rangle[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle & \equiv_\lambda & (M[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle)[\widetilde{y} \leftarrow y]\langle\!\langle A/y \rangle\!\rangle,\ \begin{array}{r} x_i \in \widetilde{x} \Rightarrow x_i \notin \mathsf{fv}(A), \\ y_i \in \widetilde{y} \Rightarrow y_i \notin \mathsf{fv}(B) \end{array} \\
M\langle\!\lvert N_2/y \rvert\!\rangle\langle\!\lvert N_1/x \rvert\!\rangle & \equiv_\lambda & M\langle\!\lvert N_1/x \rvert\!\rangle\langle\!\lvert N_2/y \rvert\!\rangle, x \notin \mathsf{fv}(N_2), y \notin \mathsf{fv}(N_1) \\
M\llbracket U_2/y \rrbracket\llbracket U_1/x \rrbracket & \equiv_\lambda & M\llbracket U_1/x \rrbracket\llbracket U_2/y \rrbracket, x \notin \mathsf{fv}(U_2), y \notin \mathsf{fv}(U_1) \\
C[M] & \equiv_\lambda & C[M'],\ \text{with } M \equiv_\lambda M' \\
D[\mathbb{M}] & \equiv_\lambda & D[\mathbb{M}'],\ \text{with } \mathbb{M} \equiv_\lambda \mathbb{M}'
\end{array}
$$

The first rule states that we may remove unneeded unrestricted substitutions when the variable in concern does not appear within the term. The next three identities enforce that bags can always be moved in and out of all forms of explicit substitution, which are useful manipulate expressions and to form a redex for Rule [$\mathtt{R:Beta}$]. The other rules deal with permutation of explicit substitutions and contextual closure.

Well-formedness for $u\widehat{\lambda}^{\lightning}_{\oplus}$, based on intersection types, is defined as in § 3; see App. D.

4.3 Encoding $u\lambda^{\lightning}_{\oplus}$ into $u\widehat{\lambda}^{\lightning}_{\oplus}$

We define an encoding $(\!\lvert \cdot \rvert\!)^{\circ}$ from well-formed terms in $u\lambda^{\lightning}_{\oplus}$ into $u\widehat{\lambda}^{\lightning}_{\oplus}$. This encoding relies on an intermediate encoding $(\!\lvert \cdot \rvert\!)^{\bullet}$ on $u\lambda^{\lightning}_{\oplus}$-terms.

$$
\begin{array}{ccc}
\llparenthesis x \rrparenthesis^{\bullet} = x & \llparenthesis x[i] \rrparenthesis^{\bullet} = x[i] & \llparenthesis \mathtt{1} \rrparenthesis^{\bullet} = \mathtt{1} \\
\llparenthesis \mathtt{1}^{!} \rrparenthesis^{\bullet} = \mathtt{1}^{!} & \llparenthesis \mathtt{fail}^{\widetilde{x}} \rrparenthesis^{\bullet} = \mathtt{fail}^{\widetilde{x}} & \llparenthesis M\ B \rrparenthesis^{\bullet} = \llparenthesis M \rrparenthesis^{\bullet}\ \llparenthesis B \rrparenthesis^{\bullet} \\
\llparenthesis \wr M \int^{!} \rrparenthesis^{\bullet} = \wr M \int^{!} & \llparenthesis \wr M \int \cdot C \rrparenthesis^{\bullet} = \wr \llparenthesis M \rrparenthesis^{\bullet} \int \cdot \llparenthesis C \rrparenthesis^{\bullet} & \llparenthesis C \star U \rrparenthesis^{\bullet} = \llparenthesis C \rrparenthesis^{\bullet} \star \llparenthesis U \rrparenthesis^{\bullet} \\
\llparenthesis U \diamond V \rrparenthesis^{\bullet} = U \diamond V & \llparenthesis M \langle\!| N/x |\!\rangle \rrparenthesis^{\bullet} = \llparenthesis M \rrparenthesis^{\bullet} \langle\!| \llparenthesis N \rrparenthesis^{\bullet}/x |\!\rangle & \llparenthesis M \llbracket U/x \rrbracket \rrparenthesis^{\bullet} = \llparenthesis M \rrparenthesis^{\bullet} \llbracket \llparenthesis U \rrparenthesis^{\bullet}/x \rrbracket
\end{array}
$$

$$
\llparenthesis \lambda x.M \rrparenthesis^{\bullet} = \lambda x.(\llparenthesis M\langle x_1, \cdots, x_n/x\rangle \rrparenthesis^{\bullet}[x_1, \cdots, x_n \leftarrow x]) \qquad \#(x, M) = n, \text{ each } x_i \text{ is fresh}
$$

$$
\llparenthesis M \langle\!\langle C \star U/x \rangle\!\rangle \rrparenthesis^{\bullet} = \begin{cases} \displaystyle\sum_{C_i \in \mathsf{PER}(\llparenthesis C \rrparenthesis^{\bullet})} \llparenthesis M\langle \widetilde{x}/x\rangle \rrparenthesis^{\bullet} \langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket, & \text{if } \#(x, M) = \mathsf{size}(C) = k \\ \llparenthesis M\langle x_1, \cdots, x_k/x\rangle \rrparenthesis^{\bullet}[x_1, \cdots, x_k \leftarrow x]\langle\!\langle \llparenthesis C \star U \rrparenthesis^{\bullet}/x \rangle\!\rangle, & \text{if } \#(x, M) = k \geq 0 \end{cases}
$$

Figure 7 Auxiliary Encoding: $u\lambda^{\lightning}_{\oplus}$ into $u\widehat{\lambda}^{\lightning}_{\oplus}$

▶ **Notation 5.** *Given a term M such that $\#(x, M) = k$ and a sequence of pairwise distinct fresh variables $\widetilde{x} = x_1, \ldots, x_k$ we write $M\langle \widetilde{x}/x\rangle$ or $M\langle x_1, \cdots, x_k/x\rangle$ to stand for $M\langle x_1/x\rangle \cdots \langle x_k/x\rangle$, i.e., a simultaneous linear substitution whereby each distinct linear occurrence of x in M is replaced by a distinct $x_i \in \widetilde{x}$. Notice that each x_i has the same type as x. We use (simultaneous) linear substitutions to force all bound linear variables in $u\lambda^{\lightning}_{\oplus}$ to become shared variables in $u\widehat{\lambda}^{\lightning}_{\oplus}$.*

▶ **Definition 24** (From $u\lambda^{\lightning}_{\oplus}$ to $u\widehat{\lambda}^{\lightning}_{\oplus}$). *Let $M \in u\lambda^{\lightning}_{\oplus}$. Suppose $\Theta; \Gamma \models M : \tau$, with $\mathtt{dom}(\Gamma) = \mathsf{lfv}(M) = \{x_1, \cdots, x_k\}$ and $\#(x_i, M) = j_i$. We define $\llparenthesis M \rrparenthesis^{\circ}$ as*

$$
\llparenthesis M \rrparenthesis^{\circ} = \llparenthesis M\langle \widetilde{x}_1/x_1\rangle \cdots \langle \widetilde{x}_k/x_k\rangle \rrparenthesis^{\bullet}[\widetilde{x}_1 \leftarrow x_1] \cdots [\widetilde{x}_k \leftarrow x_k]
$$

where $\widetilde{x}_i = x_{i_1}, \cdots, x_{j_i}$ and the encoding $\llparenthesis \cdot \rrparenthesis^{\bullet} : u\lambda^{\lightning}_{\oplus} \to u\widehat{\lambda}^{\lightning}_{\oplus}$ is defined in Fig. 7 on $u\lambda^{\lightning}_{\oplus}$-terms. The encoding $\llparenthesis \cdot \rrparenthesis^{\circ}$ extends homomorphically to expressions.

The encoding $\llparenthesis \cdot \rrparenthesis^{\circ}$ converts n occurrences of x in a term into n distinct variables $x_1, \ldots, x_n$. The sharing construct coordinates them by constraining each to occur exactly once within a term. We proceed in two stages. First, we share all linear free linear variables using $\llparenthesis \cdot \rrparenthesis^{\circ}$: this ensures that free variables are replaced by shared variables which are then bound by the sharing construct. Second, we apply the encoding $\llparenthesis \cdot \rrparenthesis^{\bullet}$ on the corresponding term. The encoding is presented in Fig. 7: $\llparenthesis \cdot \rrparenthesis^{\bullet}$ maintains $x[i]$ unaltered, and acts homomorphically over concatenation of bags and explicit substitutions. The encoding renames bound variables with bound shared variables. As we will see, this will enable a tight operational correspondence result with $\mathsf{s}\pi$. In App. E we establish the correctness of $\llparenthesis \cdot \rrparenthesis^{\circ}$.

▶ **Example 25.** We apply the encoding $\llparenthesis \cdot \rrparenthesis^{\bullet}$ in some of the $u\lambda^{\lightning}_{\oplus}$-terms from Example 3: for simplicity, we assume that N and U have no free variables.

$$
\llparenthesis (\lambda x.x)\ \wr N \int \star U \rrparenthesis^{\bullet} = \llparenthesis \lambda x.x \rrparenthesis^{\bullet} \llparenthesis \wr N \int \star U \rrparenthesis^{\bullet} = \lambda x.x_1[x_1 \leftarrow x]\ \wr \llparenthesis N \rrparenthesis^{\bullet} \int \star \llparenthesis U \rrparenthesis^{\bullet}
$$

$$
\llparenthesis (\lambda x.x[1])\mathtt{1} \star \wr N \int^{!} \diamond U \rrparenthesis^{\bullet} = \llparenthesis (\lambda x.x[1] \rrparenthesis^{\bullet} \llparenthesis \mathtt{1} \star \wr N \int^{!} \diamond U \rrparenthesis^{\bullet} = (\lambda x.x[1][\leftarrow x])\mathtt{1} \star \wr \llparenthesis N \rrparenthesis^{\bullet} \int^{!} \diamond \llparenthesis U \rrparenthesis^{\bullet}
$$

4.4 Encoding $u\widehat{\lambda}^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$

We now define our encoding of $u\widehat{\lambda}^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$, and establish its correctness.

▶ **Notation 6.** *To help illustrate the behaviour of the encoding, we use the names x, x^{ℓ}, and $x^{!}$ to denote three distinct channel names: while x^{ℓ} is the channel that performs the linear substitution behaviour of the encoded term, channel $x^{!}$ performs the unrestricted behaviour.*

▶ **Definition 26** (From $u\widehat{\lambda}^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$: Expressions)**.** *Let u be a name. The encoding $[\![\cdot]\!]_u : u\widehat{\lambda}^{\lightning}_{\oplus} \to \mathsf{s}\pi$ is defined in Fig. 8.*

Every (free) variable x in an $u\widehat{\lambda}^{\lightning}_{\oplus}$ expression becomes a name x in its corresponding $\mathsf{s}\pi$ process. As customary in encodings of λ into π, we use a name u to provide the behaviour of the encoded expression. In our case, u is a non-deterministic session: the encoded expression can be effectively available or not; this is signalled by prefixes $u.\overline{\texttt{some}}$ and $u.\overline{\texttt{none}}$, respectively.

We discuss the most salient aspects of the encoding in Fig. 8.

- While linear variables are encoded as in [17], the encoding of an unrestricted variable $x[j]$, not treated in [17], is much more interesting: it first connects to a server along channel x via a request $\overline{x^!}?(x_i)$ followed by a selection on $x_i.l_j$, which takes the j-th branch.
- The encoding of $\lambda x.M[\widetilde{x} \leftarrow x]$ confirms its behaviour first followed by the receiving of a channel x. The channel x provides a linear channel x^ℓ and an unrestricted channel $x^!$ for dedicated substitutions of the linear and unrestricted bag components.
- We encode $M\,(C \star U)$ as a non-deterministic sum: an application involves a choice in the order in which the elements of C are substituted.
- The encoding of $C \star U$ synchronises with the encoding of $\lambda x.M[\widetilde{x} \leftarrow x]$. The channel x^ℓ provides the linear behaviour of the bag C while $x^!$ provides the behaviour of U; this is done by guarding the encoding of U with a server connection such that every time a channel synchronises with $!x^!(x_i)$ a fresh copy of U is spawned.
- The encoding of $\wr M \int \cdot C$ synchronises with the encoding of $M[\widetilde{x} \leftarrow x]$, just discussed. The name y_i is used to trigger a failure in the computation if there is a lack of elements in the encoding of the bag.
- The encoding of $M[\widetilde{x} \leftarrow x]$ first confirms the availability of the linear behaviour along x^ℓ. Then it sends a name y_i, which is used to collapse the process in the case of a failed reduction. Subsequently, for each shared variable, the encoding receives a name, which will act as an occurrence of the shared variable. At the end, a failure prefix on x is used to signal that there is no further information to send over.
- The encoding of U synchronises with the last half encoding of $x[j]$; the name x_i selects the j-th term in the unrestricted bag.
- The encoding of $M\langle\!|N/x|\!\rangle$ is the composition of the encodings of M and N, where we await a confirmation of a behaviour along the variable that is being substituted.
- $M\|U/x\|$ is encoded as the composition of the encoding of M and a server guarding the encoding of U: in order for $[\![M]\!]_u$ to gain access to $[\![U]\!]_{x_i}$ it must first synchronise with the server channel $x^!$ to spawn a fresh copy of U.
- The encoding of $\mathbb{M} + \mathbb{N}$ homomorphically preserves non-determinism. Finally, the encoding of $\texttt{fail}^{x_1,\cdots,x_k}$ simply triggers failure on u and on each of $x_1, \cdots, x_k$.

▶ **Example 27.** [Cont. Example 3] We illustrate the encoding $[\![\cdot]\!]$ on the $u\widehat{\lambda}^{\lightning}_{\oplus}$-terms/bags occurring in $M_1 = \lambda x.x_1[x_1 \leftarrow x](\wr (\!|N|\!)^{\bullet} \int \star (\!|U|\!)^{\bullet})$ as below:

$$[\![\lambda x.x_1[x_1 \leftarrow x]]\!]_v = v.\overline{\texttt{some}}; v(x).x.\overline{\texttt{some}}; x(x^\ell).x(x^!).x[\,]; [\![x_1[x_1 \leftarrow x]]\!]_v$$

$$[\![\wr (\!|N|\!)^{\bullet} \int \star (\!|U|\!)^{\bullet}]\!]_x = x.\texttt{some}_{\mathsf{lfv}(\wr (\!|N|\!)^{\bullet} \int)}; \overline{x}(x^\ell).([\![(\!|N|\!)^{\bullet}]\!]_{x^\ell} \mid \overline{x}(x^!).(!x^!(x_i).[\![(\!|U|\!)^{\bullet}]\!]_{x_i} \mid \overline{x}[\,]))$$

$$
\begin{aligned}
[\![x]\!]_u &= x.\overline{\mathtt{some}}; [x \leftrightarrow u] \\
[\![x[j]]\!]_u &= \overline{x^!}?(x_i).x_i.l_j; [x_i \leftrightarrow u] \\
[\![\lambda x.M[\widetilde{x} \leftarrow x]]\!]_u &= u.\overline{\mathtt{some}}; u(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \\
[\![M[\widetilde{x} \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle]\!]_u &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \mid [\![C_i \star U]\!]_x) \\
[\![M(C \star U)]\!]_u &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \\
[\![C \star U]\!]_x &= x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^\ell).\big([\![C]\!]_{x^\ell} \mid \overline{x}(x^!).(!x^!(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}})\big) \\
[\![\wr M \int \cdot C]\!]_{x^\ell} &= x^\ell.\mathtt{some}_{\mathsf{lfv}(\wr M \int \cdot C)}; x^\ell(y_i).x^\ell.\mathtt{some}_{y_i,\mathsf{lfv}(\wr M \int \cdot C)}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_i). \\
&\qquad (x_i.\mathtt{some}_{\mathsf{lfv}(M)}; [\![M]\!]_{x_i} \mid [\![C]\!]_{x^\ell} \mid y_i.\overline{\mathtt{none}}) \\
[\![1]\!]_{x^\ell} &= x^\ell.\mathtt{some}_\emptyset; x^\ell(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \\
[\![1^!]\!]_x &= x.\overline{\mathtt{none}} \\
[\![\wr N \int^!]\!]_x &= [\![N]\!]_x \\
[\![U]\!]_x &= x.\mathtt{case}_{U_i \in U}\{\mathtt{l}_i : [\![U_i]\!]_x\} \\
[\![M \langle\!| N/x |\!\rangle]\!]_u &= (\nu x)([\![M]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x) \\
[\![M \llbracket U/x \rrbracket]\!]_u &= (\nu x^!)([\![M]\!]_u \mid \ !x^!(x_i).[\![U]\!]_{x_i}) \\
[\![M[\leftarrow x]]\!]_u &= x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_i).(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; [\![M]\!]_u \mid x^\ell.\overline{\mathtt{none}}) \\
[\![M[x_1, \cdots, x_n \leftarrow x]]\!]_u &= x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_1).\big(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; \mathbf{0} \mid \\
&\qquad x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_1,\cdots,x_n)}; x^\ell(x_1).[\![M[x_2, \cdots, x_n \leftarrow x]]\!]_u\big) \\
[\![\mathbb{M} + \mathbb{N}]\!]_u &= [\![\mathbb{M}]\!]_u \oplus [\![\mathbb{N}]\!]_u \\
[\![\mathtt{fail}^{x_1,\cdots,x_k}]\!]_u &= u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid \cdots \mid x_k.\overline{\mathtt{none}}
\end{aligned}
$$

Figure 8 Encoding $u\widehat{\lambda}^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$ (cf. Def. 26).

$$
\begin{aligned}
[\![(\!|M_1|\!)^\bullet]\!]_u &= [\![\lambda x.x_1[x_1 \leftarrow x] \wr (\!|N|\!)^\bullet \int \star (\!|U|\!)^\bullet]\!]_u \\
&= (\nu v)([\![\lambda x.x_1[x_1 \leftarrow x]]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}((\!|N|\!)^\bullet)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![\wr (\!|N|\!)^\bullet \int \star (\!|U|\!)^\bullet]\!]_x)) \\
&= (\nu v)(v.\overline{\mathtt{some}}; v(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x[\,]; x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_1).(y_1.\mathtt{some}_\emptyset; y_1[\,]; \mathbf{0} \mid \\
&\quad x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_u; x^\ell(x_1).x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_2).(y_2.\mathtt{some}_{u,x_1}; y_2[\,]; [\![x_1]\!]_v \mid x^\ell.\overline{\mathtt{none}})) \mid \\
&\quad v.\mathtt{some}_{u,\mathsf{lfv}((\!|N|\!)^\bullet)}; \overline{v}(x).([v \leftrightarrow u] \mid \\
&\quad x.\mathtt{some}_{\mathsf{lfv}((\!|N|\!)^\bullet)}; \overline{x}(x^\ell).(x^\ell.\mathtt{some}_{\mathsf{lfv}((\!|N|\!)^\bullet)}; x^\ell.\mathtt{some}_{y_1,\mathsf{lfv}(\wr (\!|N|\!)^\bullet \int)}; \\
&\quad x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_1).(x_1.\mathtt{some}_{\mathsf{lfv}((\!|N|\!)^\bullet)}; [\![(\!|N|\!)^\bullet]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell(y_2). \\
&\quad (y_2.\overline{\mathtt{some}}; \overline{y_2}[\,] \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}})) \mid \overline{x}(x^!).(!x^!(x_i).[\![(\!|U|\!)^\bullet]\!]_{x_i} \mid \overline{x}[\,]))))
\end{aligned}
$$

We now encode intersection types (for $u\widehat{\lambda}^{\lightning}_{\oplus}$) into session types (for $\mathsf{s}\pi$).

▶ **Definition 28** (From $u\widehat{\lambda}^{\lightning}_{\oplus}$ into $\mathsf{s}\pi$: Types)**.** *The translation $[\![\cdot]\!]$ in Figure 9 extends as follows to a context $\Gamma = x_1{:}\sigma_1, \cdots, x_m{:}\sigma_m, v_1{:}\pi_1, \cdots, v_n{:}\pi_n$ and a context $\Theta = x^!_1{:}\eta_1, \cdots, x^!_k{:}\eta_k$:*

$$
\begin{aligned}
[\![\Gamma]\!] &= x_1 : \&\overline{[\![\sigma_1]\!]}, \cdots, x_m : \&\overline{[\![\sigma_m]\!]}, v_1 : \overline{[\![\pi_1]\!]_{(\sigma,i_1)}}, \cdots, v_n : \overline{[\![\pi_n]\!]_{(\sigma,i_n)}} \\
[\![\Theta]\!] &= x^!_1 : \overline{[\![\eta_1]\!]}, \cdots, x^!_k : \overline{[\![\eta_k]\!]}
\end{aligned}
$$

This encoding formally expresses how non-deterministic session protocols (typed with '&') capture linear and unrestricted resource consumption in $u\widehat{\lambda}^{\lightning}_{\oplus}$. Notice that the encoding of the multiset type π depends on two arguments (a strict type σ and a number $i \geq 0$) which are left unspecified above. This is crucial to represent failures in $u\widehat{\lambda}^{\lightning}_{\oplus}$ as typable processes in $\mathsf{s}\pi$.

$$
\begin{aligned}
[\![\mathbf{unit}]\!] &= \&\mathbf{1} \\
[\![\eta]\!] &= \&_{\eta_i \in \eta}\{\mathtt{l}_i ; [\![\eta_i]\!]\} \\
[\![(\sigma^k, \eta) \to \tau]\!] &= \&(\overline{[\![(\sigma^k, \eta)]\!]_{(\sigma,i)}} \parr [\![\tau]\!]) \\
[\![(\sigma^k, \eta)]\!]_{(\sigma,i)} &= \oplus(([\![\sigma^k]\!]_{(\sigma,i)}) \otimes ((![\![\eta]\!]) \otimes (\mathbf{1}))) \\
[\![\sigma \wedge \pi]\!]_{(\sigma,i)} &= \overline{\&((\oplus\bot) \otimes (\& \oplus ((\&\overline{[\![\sigma]\!]}) \parr (\overline{[\![\pi]\!]_{(\sigma,i)}}))))} \\
&= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\pi]\!]_{(\sigma,i)})))) \\
[\![\omega]\!]_{(\sigma,i)} &= \begin{cases} \overline{\&((\oplus\bot) \otimes (\& \oplus \bot)))} & \text{if } i = 0 \\ \overline{\&((\oplus\bot) \otimes (\& \oplus ((\&\overline{[\![\sigma]\!]}) \parr (\overline{[\![\omega]\!]_{(\sigma,i-1)}}))))} & \text{if } i > 0 \end{cases}
\end{aligned}
$$

Figure 9 Encoding of intersection types into session types (cf. Def. 28)

For instance, given $(\sigma^j, \eta) \to \tau$ and (σ^k, η), the well-formedness rule for application admits a mismatch ($j \neq k$, cf. Rule [`FS:app`] in Fig. 14, App. D). In our proof of type preservation, the two arguments of the encoding are instantiated appropriately. Notice also how the client-server behaviour of unrestricted resources appears as '$![\![\eta]\!]$' in the encoding of the tuple type (σ^k, η). With our encodings of expressions and types in place, we can now define our encoding of judgements:

▶ **Definition 29.** *If $\mathbb{M}$ is an $u\widehat{\lambda}^{\lightning}_{\oplus}$ expression such that $\Theta; \Gamma \models \mathbb{M} : \tau$ then we define the encoding of the judgement to be: $[\![\mathbb{M}]\!]_u \vdash [\![\Gamma]\!], u : [\![\tau]\!]; [\![\Theta]\!]$.*

The correctness of our encoding $[\![\cdot]\!]_u : u\lambda^{\lightning}_{\oplus} \to \mathsf{s}\pi$, stated in Theorem 31 (and detailed in App. F), relies on a notion of *success* for both $u\lambda^{\lightning}_{\oplus}$ and $\mathsf{s}\pi$, given by the $\checkmark$ construct:

▶ **Definition 30.** *We extend the syntax of terms for $u\widehat{\lambda}^{\lightning}_{\oplus}$ and processes for $\mathsf{s}\pi$ with $\checkmark$:*

- **(In $u\widehat{\lambda}^{\lightning}_{\oplus}$)** *$\mathbb{M} \Downarrow_{\checkmark}$ iff there exist $M_1, \cdots, M_k$ such that $\mathbb{M} \longrightarrow^* M_1 + \cdots + M_k$ and $\mathsf{head}(M'_j) = \checkmark$, for some $j \in \{1, \ldots, k\}$ and term M'_j such that $M_j \equiv_\lambda M'_j$.*
- **(In $\mathsf{s}\pi$)** *$P \Downarrow_{\checkmark}$ holds whenever there exists a P' such that $P \longrightarrow^* P'$ and P' contains an unguarded occurrence of $\checkmark$ (i.e., an occurrence that does not occur behind a prefix).*

We now state operational correctness. Fig. 10 illustrates the relation between completeness and soundness that the encoding satisfies: solid arrows denote reductions assumed, dashed arrows denote the application of $[\![\cdot]\!]_u$, and dotted arrows denote the existing reductions that can be implied from the results.

We remark that since $u\widehat{\lambda}^{\lightning}_{\oplus}$ satisfies the diamond property, it suffices to consider completeness based on a single reduction ($\mathbb{N} \longrightarrow \mathbb{M}$). Soundness uses the congruence $\equiv_\lambda$ in Def. 23. We write $N \longrightarrow_{\equiv_\lambda} N'$ iff $N \equiv_\lambda N_1 \longrightarrow N_2 \equiv_\lambda N'$, for some N_1, N_2. Then, $\longrightarrow^*_{\equiv_\lambda}$ is the reflexive, transitive closure of $\longrightarrow_{\equiv_\lambda}$. For success sensitivity, we decree $[\![\checkmark]\!]_u = \checkmark$. We have:

▶ **Theorem 31** (Operational Correctness)**.** *Let $\mathbb{N}$ and $\mathbb{M}$ be well-formed $u\widehat{\lambda}^{\lightning}_{\oplus}$ closed expressions.*

(a) *(Type Preservation) Let B be a bag. We have:*
  - **(i)** *If $\Theta; \Gamma \models B : (\sigma^k, \eta)$ then $[\![B]\!]_u \models [\![\Gamma]\!], u : [\![(\sigma^k, \eta)]\!]_{(\sigma,i)}; [\![\Theta]\!]$.*
  - **(ii)** *If $\Theta; \Gamma \models \mathbb{M} : \tau$ then $[\![\mathbb{M}]\!]_u \models [\![\Gamma]\!], u : [\![\tau]\!]; [\![\Theta]\!]$.*

(b) *(Completeness) If $\mathbb{N} \longrightarrow \mathbb{M}$ then there exists Q such that $[\![\mathbb{N}]\!]_u \longrightarrow^* Q \equiv_\lambda [\![\mathbb{M}]\!]_u$.*

(c) *(Soundness) If $[\![\mathbb{N}]\!]_u \longrightarrow^* Q$ then $Q \longrightarrow^* Q'$, $\mathbb{N} \longrightarrow^*_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u \equiv Q'$, for some $Q', \mathbb{N}'$.*

(d) *(Success Sensitivity) $\mathbb{M} \Downarrow_{\checkmark}$ if, and only if, $[\![\mathbb{M}]\!]_u \Downarrow_{\checkmark}$.*

Proof. Below we illustrate the most interesting case of the proof of soundness. Detailed proof can be found in App. F. ◀

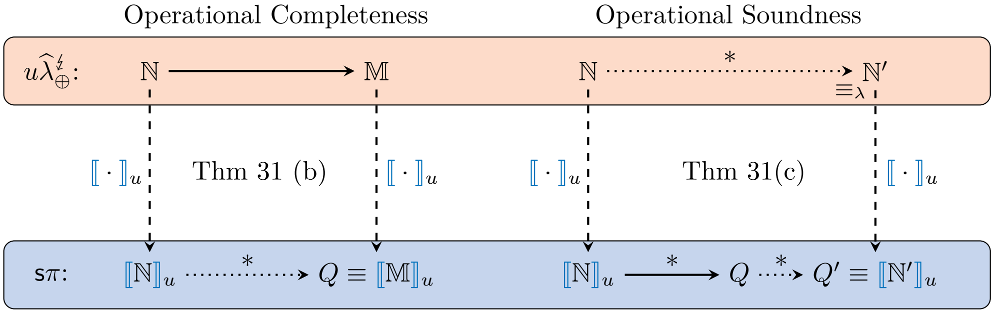

Figure 10 An overview of operational soundness and completeness for $[\![\cdot]\!]_u$.

Proof. All items are proven by structural induction; a detailed proof can be found in App. F. Below we present the most interesting case in the proof of *soundness*: the case when $\mathbb{N} = M(C \star U)$. Then,

$$[\![\mathbb{N}]\!]_u = [\![M(C \star U)]\!]_u = \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)).$$

The proof then proceeds by induction on the number of reduction steps k that can be taken from $[\![\mathbb{N}]\!]_u$, i.e, $[\![\mathbb{N}]\!]_u \longrightarrow^k Q$. We will consider the case when $k \geq 1$, where for some process R and non-negative integers n, m such that $k = n + m$, we have the following:

$$[\![\mathbb{N}]\!]_u \longrightarrow^m \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(R \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \longrightarrow^n Q$$

There are several cases to analyse depending on the values of m and n, and the shape of M. We consider $m = 0$, $n \geq 1$ and $M = (\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!\langle N_1/y_1 \rangle\!\rangle \cdots \langle\!\langle N_p/y_p \rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket$, where $p, q \geq 0$. Then, $[\![\mathbb{N}]\!]_u$ can perform the following reduction:

$$[\![\mathbb{N}]\!]_u \longrightarrow^* \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu \widetilde{y}, \widetilde{z}, x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x[\,]; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \mid Q'' \mid [\![C_i \star U]\!]_x) \;(:= Q_3)$$

where Q'' defines the encoding of explicit substitutions within the encoded subterm M. Notice that:

$$\begin{aligned}
\mathbb{N} &= (\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!\langle N_1/y_1 \rangle\!\rangle \cdots \langle\!\langle N_p/y_p \rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket (C \star U) \\
&\equiv_\lambda (\lambda x.(M'[\widetilde{x} \leftarrow x])(C \star U))\langle\!\langle N_1/y_1 \rangle\!\rangle \cdots \langle\!\langle N_p/y_p \rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket \\
&\longrightarrow M'[\widetilde{x} \leftarrow x]\langle\!\langle (C \star U)/x \rangle\!\rangle \langle\!\langle N_1/y_1 \rangle\!\rangle \cdots \langle\!\langle N_p/y_p \rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket = \mathbb{M}
\end{aligned}$$

where the congruence holds assuming the necessary α-renaming of variables. Finally, one can verify that $[\![\mathbb{M}]\!]_u = Q_3$, and the result follows. ◀

▶ **Example 32.** Recall again term M_1 from Example 3. It can be shown that $(\!|M_1|\!)^\bullet \longrightarrow^* (\!|N|\!)^\bullet \llbracket (\!|U|\!)^\bullet / x^! \rrbracket$. To illustrate operational completeness, we can verify preservation of reduction, via $[\![\cdot]\!]$: reductions below use the rules for $\mathsf{s}\pi$ in Figure 5—see Figure 11.

5 Concluding Remarks

Summary We have extended the line of work we developed in [17], on resource λ-calculi with firm logical foundations via typed concurrent processes. We presented $u\lambda^{\not{\,}}_{\oplus}$, a resource

$$
\begin{aligned}
&\llbracket \llparenthesis M_1 \rrparenthesis^{\bullet} \rrbracket = \\
&(\nu v)(v.\overline{\texttt{some}}; v(x).x.\overline{\texttt{some}}; x(x^{\ell}).x(x^{!}).x[\,]; x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_1).(y_1.\texttt{some}_{\emptyset}; y_1[\,]; \mathbf{0} \mid \\
&\quad x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; x^{\ell}(x_1).x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_2).(y_2.\texttt{some}_{u,x_1}; y_2.[\,]; \llbracket x_1 \rrbracket_v \mid x^{\ell}.\overline{\texttt{none}})) \mid \\
&\quad v.\texttt{some}_{u,\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \overline{v}(x).([v \leftrightarrow u] \mid x.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \overline{x}(x^{\ell}).(x^{\ell}.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}(y_1). \\
&\quad x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\wr \llparenthesis N \rrparenthesis^{\bullet} \int)}; x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid y_1.\overline{\texttt{none}} \mid x^{\ell}.\texttt{some}_{\emptyset}; \\
&\quad x^{\ell}(y_2).(y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid \overline{x}(x^{!}).(!x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i} \mid \overline{x}[\,])))) \\
&\longrightarrow^{3} (\nu x)(x.\overline{\texttt{some}}; x(x^{\ell}).x(x^{!}).x[\,]; x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_1).(y_1.\texttt{some}_{\emptyset}; y_1[\,]; \mathbf{0} \mid x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; \\
&\quad x^{\ell}(x_1).x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_2).(y_2.\texttt{some}_{u,x_1}; y_2[\,]; \llbracket x_1 \rrbracket_u \mid x^{\ell}.\overline{\texttt{none}})) \mid (x.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \overline{x}(x^{\ell}). \\
&\quad (x^{\ell}.\texttt{some}_{\mathsf{lfv}(\wr \llparenthesis N \rrparenthesis^{\bullet} \int)}; x^{\ell}(y_1).x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\wr \llparenthesis N \rrparenthesis^{\bullet} \int)}; x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid \\
&\quad y_1.\overline{\texttt{none}} \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}(y_2).(y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid \overline{x}(x^{!}).(!x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i} \mid \overline{x}[\,])))) \\
&\longrightarrow^{2} (\nu x, x^{\ell})(x(x^{!}).x.[\,]; x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_1).(y_1.\texttt{some}_{\emptyset}; y_1[\,]; \mathbf{0} \mid x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; x^{\ell}(x_1). \\
&\quad x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_2).(y_2.\texttt{some}_{u,x_1}; y_2[\,]; \llbracket x_1 \rrbracket_u \mid x^{\ell}.\overline{\texttt{none}})) \mid (x^{\ell}.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}(y_1). \\
&\quad x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid y_1.\overline{\texttt{none}} \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}(y_2). \\
&\quad (y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid \overline{x}(x^{!}).(!x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i} \mid \overline{x}[\,])))) \\
&\longrightarrow (\nu x, x^{\ell}, x^{!})(x[\,]; x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_1).(y_1.\texttt{some}_{\emptyset}; y_1[\,]; \mathbf{0} \mid x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; x^{\ell}(x_1).x^{\ell}.\overline{\texttt{some}}. \\
&\quad \overline{x^{\ell}}(y_2).(y_2.\texttt{some}_{u,x_1}; y_2[\,]; \llbracket x_1 \rrbracket_u \mid x^{\ell}.\overline{\texttt{none}})) \mid (x^{\ell}.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}(y_1).x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \\
&\quad x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid y_1.\overline{\texttt{none}} \mid \\
&\quad x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}(y_2).(y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid !x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i} \mid \overline{x}[\,]))) \\
&\longrightarrow (\nu x^{\ell}, x^{!})(x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_1).(y_1.\texttt{some}_{\emptyset}; y_1.[\,]; \mathbf{0} \mid x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; x^{\ell}(x_1). \\
&\quad x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_2).(y_2.\texttt{some}_{u,x_1}; y_2.[\,]; \llbracket x_1 \rrbracket_u \mid x^{\ell}.\overline{\texttt{none}})) \mid (x^{\ell}.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}(y_1). \\
&\quad x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid y_1.\overline{\texttt{none}} \mid \\
&\quad x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}(y_2).(y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid !x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i}))) \\
&\longrightarrow (\nu x^{\ell}, y_1, x^{!})(y_1.\texttt{some}_{\emptyset}; y_1[\,]; \mathbf{0} \mid x^{\ell}.\overline{\texttt{some}}; x^{\ell}.\texttt{some}_{u}; x^{\ell}(x_1).x^{\ell}.\overline{\texttt{some}}.\overline{x^{\ell}}(y_2). \\
&\quad (y_2.\texttt{some}_{u,x_1}; y_2[\,]; \llbracket x_1 \rrbracket_u \mid x^{\ell}.\overline{\texttt{none}}) \mid (x^{\ell}.\texttt{some}_{y_1,\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; x^{\ell}.\overline{\texttt{some}}; \overline{x^{\ell}}(x_1).(x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \\
&\quad \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid y_1.\overline{\texttt{none}} \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}(y_2).(y_2.\overline{\texttt{some}}; \overline{y_2}[\,] \mid x^{\ell}.\texttt{some}_{\emptyset}; x^{\ell}.\overline{\texttt{none}})) \mid !x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i})) \\
&\longrightarrow^{*} (\nu x_1, x^{!})(x_1.\overline{\texttt{some}}; [x_1 \leftrightarrow u] \mid x_1.\texttt{some}_{\mathsf{lfv}(\llparenthesis N \rrparenthesis^{\bullet})}; \llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_{x_1} \mid !x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i}) \\
&\longrightarrow^{*} (\nu x^{!})(\llbracket \llparenthesis N \rrparenthesis^{\bullet} \rrbracket_u \mid !x^{!}(x_i).\llbracket \llparenthesis U \rrparenthesis^{\bullet} \rrbracket_{x_i}) \\
&= \llbracket \llparenthesis N \rrparenthesis^{\bullet} \llfloor \llparenthesis U \rrparenthesis^{\bullet} / x^{!} \rrceil \rrbracket_u
\end{aligned}
$$

Figure 11 Illustrating operational correspondence, following Example 32.

calculus with non-determinism and explicit failures, with dedicated treatment for linear and unrestricted resources. By means of examples, we illustrated the expressivity, (lazy) semantics, and design decisions underpinning $u\lambda^{\lightning}_{\oplus}$, and introduced a class of well-formed expressions based on intersection types, which includes fail-prone expressions. To bear witness to the logical foundations of $u\lambda^{\lightning}_{\oplus}$, we defined and proved correct a typed encoding into the concurrent calculus $\mathsf{s}\pi$, which subsumes the one in [17]. We plan to study key properties for $u\lambda^{\lightning}_{\oplus}$ (such as solvability and normalisation) by leveraging our typed encoding into $\mathsf{s}\pi$.

Related Work With respect to previous resource calculi, a distinctive feature of $u\lambda^{\lightning}_{\oplus}$ is its support of explicit failures, which may arise depending on the interplay between (i) linear and unrestricted occurrences of variables in a term and (ii) associated resources in the bag. This feature allows $u\lambda^{\lightning}_{\oplus}$ to express variants of usual λ-terms ($\mathbf{I}$, Δ, Ω) not expressible in other resource calculi.

Related to $u\lambda^{\lightning}_{\oplus}$ is Boudol's work on a λ-calculus in which multiplicities can be infinite [1, 3]. An intersection type system is used to prove *adequacy* with respect to a testing semantics. However, failing behaviours as well as typability are not explored. Multiplicities can be expressed in $u\lambda^{\lightning}_{\oplus}$: a linear resource is available m times when the linear bag contains m copies of it; the term fails if the corresponding number of linear variables is different from m.

Also related is the resource λ-calculus by Pagani and Ronchi della Rocca [16], which includes linear and reusable resources; the latter are available in multisets, also called bags. In their setting, $M[N^!]$ denotes an application of a term M to a resource N that can be used *ad libitum*. Standard terms such as $\mathbf{I}$, Δ and Ω are expressed as $\lambda x.x$, $\Delta := \lambda x.x[x^!]$, and $\Omega := \Delta[\Delta^!]$, respectively; different variants are possible but cannot express the desired behaviour. A lazy reduction semantics is based on *baby* and *giant* steps: whereas the first consume one resource at each time, the second comprises several baby steps; combinations of the use of resources (by permuting resources in bags) are considered. A (non-idempotent) intersection type system is proposed: normalisation and a characterisation of solvability are investigated. Unlike our work, encodings into the π-calculus are not explored in [16].

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Contents

A Appendix to Section 2

A.1 Diamond Property for $u\lambda^{\lightning}_{\oplus}$

▶ **Proposition 33** (Diamond Property for $u\lambda^{\lightning}_{\oplus}$). *For all* $\mathbb{N}$, $\mathbb{N}_1$, $\mathbb{N}_2$ *in* $u\lambda^{\lightning}_{\oplus}$ *s.t.* $\mathbb{N} \longrightarrow \mathbb{N}_1$, $\mathbb{N} \longrightarrow \mathbb{N}_2$ *with* $\mathbb{N}_1 \neq \mathbb{N}_2$ *then* $\exists \mathbb{M}$ *s.t.* $\mathbb{N}_1 \longrightarrow \mathbb{M}$, $\mathbb{N}_2 \longrightarrow \mathbb{M}$.

Proof. We give a short argument to convince the reader of this. Notice that an expression can only perform a choice of reduction steps when it is a nondeterministic sum of terms in which multiple terms can perform independent reductions. For simplicity sake we will only consider an expression $\mathbb{N}$ that consist of two terms where $\mathbb{N} = N + M$. We also have that $N \longrightarrow N'$ and $M \longrightarrow M'$. Then we let $\mathbb{N}_1 = N' + M$ and $\mathbb{N}_2 = N + M'$ by the $[\mathtt{R} : \mathtt{ECont}]$ rules. Finally we prove that $\mathbb{M}$ exists by letting $\mathbb{M} = N' + M'$. ◀

B Appendix to Section 3

Here we prove subject reduction (SR) for $u\lambda^{\lightning}_{\oplus}$ (Theorem 17). It follows from two substitution lemmas: one for substituting a linear variable (Lemma 34) and another for an unrestricted variable (Lemma 35). Proofs of both lemmas are standard, by structural induction; we give a complete proof of SR in Theorem 36.

▶ **Lemma 34** (Linear Substitution Lemma for $u\lambda^{\lightning}_{\oplus}$). *If* $\Theta; \Gamma, x : \sigma \models M : \tau$, $\mathsf{head}(M) = x$, *and* $\Theta; \Delta \models N : \sigma$ *then* $\Theta; \Gamma, \Delta \models M\{\!|N/x|\!\}$.

Proof. By structural induction on M with $\mathsf{head}(M) = x$. There are three cases to be analyzed:

1. $M = x$.
   In this case, $\Theta; x : \sigma \models x : \sigma$ and $\Gamma = \emptyset$. Observe that $x\{\!|N/x|\!\} = N$, since $\Theta; \Delta \models N : \sigma$, by hypothesis, the result follows.
2. $M = M'\ B$.
   In this case, $\mathsf{head}(M'\ B) = \mathsf{head}(M') = x$, and one has the following derivation:

$$[\mathtt{F{:}app}]\ \frac{\Theta; \Gamma_1, x : \sigma \models M : (\delta^j, \eta) \to \tau \quad \Theta; \Gamma_2 \models B : (\delta^k, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, x : \sigma \models M\ B : \tau}$$

   where $\Gamma = \Gamma_1, \Gamma_2$, δ is a strict type and j, k are non-negative integers, possibly different. By IH, we get $\Theta; \Gamma_1, \Delta \models M'\{\!|N/x|\!\} : (\delta^j, \eta) \to \tau$, which gives the following derivation:

$$[\mathtt{F{:}app}]\ \frac{\Theta; \Gamma_1, \Delta \models M'\{\!|N/x|\!\} : (\delta^j, \eta) \to \tau \quad \Theta; \Gamma_2 \models B : (\delta^k, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, \Delta \models (M'\{\!|N/x|\!\})B : \tau}$$

   Therefore, from Def. 7, one has $\Theta; \Gamma_1, \Gamma_2, \Delta \models (M'\{\!|N/x|\!\})B : \tau$, and the result follows.
3. $M = M'\langle\!\langle B/y \rangle\!\rangle$.
   In this case, $\mathsf{head}(M'\langle\!\langle B/y \rangle\!\rangle) = \mathsf{head}(M') = x$, with $x \neq y$, and one has the following derivation:

$$[\mathtt{F{:}ex\text{-}sub}]\ \frac{\Theta, y^! : \eta; \Gamma_1, \hat{y} : \delta^k, x : \sigma \models M : \tau \quad \Theta; \Gamma_2 \models B : (\delta^j, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, x : \sigma \models M'\langle\!\langle B/y \rangle\!\rangle : \tau}$$

   where $\Gamma = \Gamma_1, \Gamma_2$, δ is a strict type and j, k are positive integers. By IH, we get $\Theta, y^! : \eta; \Gamma_1, \hat{y} : \delta^k, \Delta \models M'\{\!|N/x|\!\} : \tau$ and

$$[\text{F:ex-sub}]\ \frac{\Theta, y^! : \eta; \Gamma_1, \hat{y} : \delta^k, \Delta \models M'\{\!|N/x|\!\} : \tau \quad \Theta; \Gamma_2 \models B : (\delta^j, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, \Delta \models M'\{\!|N/x|\!\}\langle\!\langle B/y \rangle\!\rangle : \tau}$$

From Def. 7, $M'\langle\!\langle B/y \rangle\!\rangle\{\!|N/x|\!\} = M'\{\!|N/x|\!\}\langle\!\langle B/y \rangle\!\rangle$, therefore, $\Theta; \Gamma, \Delta \models (M'\langle\!\langle B/y \rangle\!\rangle)\{\!|N/x|\!\} : \tau$ and the result follows. ◀

▶ **Lemma 35** (Unrestricted Substitution Lemma for $u\lambda^{\lightning}_{\oplus}$). *If* $\Theta, x^! : \eta; \Gamma \models M : \tau$, $\mathsf{head}(M) = x[i]$, $\eta_i = \sigma$, *and* $\Theta; \cdot \models N : \sigma$ *then* $\Theta, x^! : \eta; \Gamma \models M\{\!|N/x[i]|\!\}$.

Proof. By structural induction on M with $\mathsf{head}(M) = x[i]$. There are three cases to be analyzed:

1. $M = x[i]$.
   In this case,

$$[\text{F:var}^!]\ \frac{[\text{F:var}^\ell]\ \dfrac{}{\Theta, x^! : \eta; x : \eta_i \models x : \sigma}}{\Theta, x^! : \eta; \cdot \models x[i] : \sigma}$$

   and $\Gamma = \emptyset$. Observe that $x[i]\{\!|N/x[i]|\!\} = N$, since $\Theta, x^! : \eta; \Gamma \models M\{\!|N/x[i]|\!\}$, by hypothesis, the result follows.
2. $M = M'\ B$.
   In this case, $\mathsf{head}(M'\ B) = \mathsf{head}(M') = x[i]$, and one has the following derivation:

$$[\text{F:app}]\ \frac{\Theta, x^! : \eta; \Gamma_1 \models M : (\delta^j, \epsilon) \to \tau \quad \Theta, x^! : \sigma; \Gamma_2 \models B : (\delta^k, \epsilon') \quad \epsilon \propto \epsilon'}{\Theta, x^! : \eta; \Gamma_1, \Gamma_2 \models M\ B : \tau}$$

   where $\Gamma = \Gamma_1, \Gamma_2$, δ is a strict type and j, k are non-negative integers, possibly different. By IH, we get $\Theta, x^! : \eta; \Gamma_1 \models M'\{\!|N/x[i]|\!\} : (\delta^j, \epsilon) \to \tau$, which gives the following derivation:

$$[\text{F:app}]\ \frac{\Theta, x^! : \eta; \Gamma_1 \models M'\{\!|N/x[i]|\!\} : (\delta^j, \epsilon) \to \tau \quad \Theta, x^! : \eta; \Gamma_2 \models B : (\delta^k, \epsilon') \quad \epsilon \propto \epsilon'}{\Theta, x^! : \eta; \Gamma_1, \Gamma_2 \models (M'\{\!|N/x[i]|\!\})B : \tau}$$

   From Def. 7, one has $\Theta, x^! : \eta; \Gamma_1, \Gamma_2 \models (M'\{\!|N/x[i]|\!\})B : \tau$, and the result follows.
3. $M = M'\langle\!\langle B/y \rangle\!\rangle$.
   In this case, $\mathsf{head}(M'\langle\!\langle B/y \rangle\!\rangle) = \mathsf{head}(M') = x[i]$, with $x \neq y$, and one has the following derivation:

$$[\text{F:ex-sub}]\ \frac{\Theta, y^! : \epsilon, x : \eta; \Gamma_1, \hat{y} : \delta^k \models M : \tau \quad \Theta, x : \eta; \Gamma_2 \models B : (\delta^j, \epsilon') \quad \epsilon \propto \epsilon'}{\Theta, x : \eta; \Gamma_1, \Gamma_2 \models M'\langle\!\langle B/y \rangle\!\rangle : \tau}$$

   where $\Gamma = \Gamma_1, \Gamma_2$, δ is a strict type and j, k are positive integers. By IH, we get $\Theta, y^! : \epsilon, x : \eta; \Gamma_1, \hat{y} : \delta^k \models M'\{\!|N/x[i]|\!\} : \tau$ and

$$[\text{F:ex-sub}]\ \frac{\Theta, y^! : \epsilon, x : \eta; \Gamma_1, \hat{y} : \delta^k, \models M'\{\!|N/x[i]|\!\} : \tau \quad \Theta, x : \sigma; \Gamma_2 \models B : (\delta^j, \epsilon') \quad \epsilon \propto \epsilon'}{\Theta, x : \eta; \Gamma_1, \Gamma_2 \models M'\{\!|N/x[i]|\!\}\langle\!\langle B/y \rangle\!\rangle : \tau}$$

   Then, $M'\langle\!\langle B/y \rangle\!\rangle\{\!|N/x[i]|\!\} = M'\{\!|N/x[i]|\!\}\langle\!\langle B/y \rangle\!\rangle$, and the result follows.

◀

▶ **Theorem 36** (SR in $u\lambda^{\lightning}_{\oplus}$). *If* $\Theta; \Gamma \models \mathbb{M} : \tau$ *and* $\mathbb{M} \longrightarrow \mathbb{M}'$ *then* $\Theta; \Gamma \models \mathbb{M}' : \tau$.

Proof. By structural induction on the reduction rules. We proceed by analysing the rule applied in $\mathbb{M}$. There are seven cases:

1. **Rule** $[\mathtt{R:Beta}]$**.**
Then $\mathbb{M} = (\lambda x.M)B \longrightarrow M\ \langle\!\langle B/x \rangle\!\rangle = \mathbb{M}'$. Since $\Theta;\Gamma \models \mathbb{M} : \tau$, one has the derivation:

$$\dfrac{[\mathtt{F{:}abs}]\ \dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau}{\Theta;\Gamma' \models \lambda x.M : (\sigma^j, \eta) \to \tau} \quad \Theta;\Delta \models B : (\sigma^k, \epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models (\lambda x.M)B : \tau}\ [\mathtt{F{:}app}]$$

for $\Gamma = \Gamma', \Delta$. Notice that

$$[\mathtt{F{:}ex\text{-}sub}]\ \dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau \quad \Theta;\Delta \models B : (\sigma^k, \epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma,\Delta \models M\langle\!\langle B/x \rangle\!\rangle : \tau}$$

Therefore, $\Theta;\Gamma \models \mathbb{M}' : \tau$ and the result follows.

2. **Rule** $[\mathtt{R:Fetch}^{\ell}]$**.**
Then $\mathbb{M} = M\ \langle\!\langle C \star U/x \rangle\!\rangle$, where $C = \wr N_1 \int \cdot \dots \cdot \wr N_k \int$, $k \geq 1$, $\#(x, M) = k$ and $\mathsf{head}(M) = x$. The reduction is as:

$$[\mathtt{R:Fetch}^{\ell}]\ \dfrac{\mathsf{head}(M) = x \quad C = \wr N_1 \int \cdot \dots \cdot \wr N_k \int,\ k \geq 1 \quad \#(x, M) = k}{M\ \langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow M\{\!|N_1/x|\!\}\langle\!\langle (C \setminus N_1) \star U/x \rangle\!\rangle + \dots + M\{\!|N_k/x|\!\}\langle\!\langle (C \setminus N_k) \star U/x \rangle\!\rangle}$$

To simplify the proof we take $k = 2$, as the case $k > 2$ is similar. Therefore, $c = \wr N_1 \int \cdot \wr N_2 \int$

$$[\mathtt{F{:}ex\text{-}sub}]\ \dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^2 \models M : \tau \quad [\mathtt{F{:}bag}]\ \dfrac{\Theta;\cdot \models U : \epsilon \quad \overset{\Pi}{\Theta;\Delta \models \wr N_1 \int \cdot \wr N_2 \int : \sigma^2}}{\Theta;\Delta \models C \star U : (\sigma^2, \epsilon)} \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models M\langle\!\langle C \star U/x \rangle\!\rangle : \tau}$$

with Π the derivation

$$[\mathtt{F{:}bag}^{\ell}]\ \dfrac{\Theta;\Delta_1 \models N_1 : \sigma \quad [\mathtt{F{:}bag}^{\ell}]\ \dfrac{\Theta;\Delta_2 \models N_2 : \sigma \quad [\mathtt{F{:}1}^{\ell}]\ \dfrac{}{\Theta;\text{-} \models 1 : \omega}}{\Theta;\Delta_2 \models \wr N_2 \int : \sigma}}{\Theta;\Delta \models \wr N_1 \int \cdot \wr N_2 \int : \sigma^2}$$

where $\Delta = \Delta_1, \Delta_2$ and $\Gamma = \Gamma', \Delta$. By Lemma 34, there exist derivations Π_1 of $\Theta, x^! : \eta; \Gamma', x : \sigma, \Delta_1 \models M\{\!|N_1/x|\!\} : \tau$ and Π_2 of $\Theta, x^! : \eta; \Gamma', x : \sigma, \Delta_2 \models M\{\!|N_2/x|\!\} : \tau$. Therefore, one has the following derivation where we omit the second case of the sum:

$$[\mathtt{F{:}sum}]\ \dfrac{[\mathtt{F{:}ex\text{-}sub}]\ \dfrac{\Pi_1 \quad [\mathtt{F{:}bag}]\ \dfrac{\Theta;\cdot \models U : \epsilon \quad \Theta;\Delta \models \wr N_2 \int : \sigma}{\Theta;\Delta \models \wr N_2 \int \star U : (\sigma, \epsilon)}}{\Theta;\Gamma',\Delta_1 \models M\{\!|N_1/x|\!\}\langle\!\langle \wr N_2 \int \star U/x \rangle\!\rangle : \tau} \quad \vdots}{\Theta;\Gamma',\Delta \models M\{\!|N_1/x|\!\}\langle\!\langle \wr N_2 \int \star U/x \rangle\!\rangle + M\{\!|N_2/x|\!\}\langle\!\langle \wr N_1 \int \star U/x \rangle\!\rangle : \tau}$$

Assuming $\mathbb{M}' = M\{\!|N_1/x|\!\}\langle\!\langle \wr N_2 \int \star B^!/x \rangle\!\rangle + M\{\!|N_2/x|\!\}\langle\!\langle \wr N_1 \int \star B^!/x \rangle\!\rangle$, the result follows.

3. **Rule** $[\mathtt{R:Fetch}^!]$**.**
Then $\mathbb{M} = M\ \langle\!\langle C \star U/x \rangle\!\rangle$, where $U = \wr N_1 \int^! \diamond \dots \diamond \wr N_l \int^!$ and $\mathsf{head}(M) = x[i]$. The reduction is as:

$$[\mathtt{R:Fetch}^!]\ \dfrac{\mathsf{head}(M) = x[i] \quad U_i = \wr N_i \int^!}{M\ \langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow M\{\!|N_i/x[i]|\!\}\langle\!\langle C \star U/x \rangle\!\rangle}$$

By hypothesis, one has the derivation:

$$\dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau \qquad [\text{F:bag}]\dfrac{\overset{\Pi}{\Theta; \cdot \models U : \epsilon} \quad \Theta; \Delta \models C : \sigma^k}{\Theta; \Delta \models C \star U : (\sigma^k, \epsilon)} \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\langle\langle C \star U / x \rangle\rangle : \tau}\ [\texttt{F:ex-sub}]$$

Where Π has the form

$$[\text{F}:\diamond\,\text{bag}^!]\dfrac{[\text{F:bag}^!]\dfrac{\Theta; \cdot \models N_1 : \epsilon_1}{\Theta; \cdot \models \wr N_1 \int^! : \epsilon_1} \quad \cdots \quad [\text{F:bag}^!]\dfrac{\Theta; \cdot \models N_l : \epsilon_l}{\Theta; \cdot \models \wr N_l \int^! : \epsilon_l}}{\Theta; \cdot \models \wr N_1 \int^! \diamond \cdots \diamond \wr N_l \int^! : \epsilon}$$

where $\Gamma = \Gamma', \Delta$. Notice that if $\epsilon_i = \delta$ and $\eta \propto \epsilon$ then $\eta_i = \delta$ By Lemma 35, there exists a derivation Π_1 of $\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!|N_i/x[i]|\!\} : \tau$. Therefore, one has the following derivation:

$$[\texttt{F:ex-sub}]\dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!|N_1/x[i]|\!\} : \tau \qquad [\text{F:bag}]\dfrac{\Theta; \cdot \models U : \epsilon \quad \Theta; \Delta \models C : \sigma^k}{\Theta; \Delta \models C \star U : (\sigma^k, \epsilon)} \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\{\!|N_i/x[i]|\!\}\langle\langle C \star U / x \rangle\rangle : \tau}$$

4. **Rule** $[\texttt{R : Fail}^\ell]$.
Then $\mathbb{M} = M\ \langle\langle C \star U / x \rangle\rangle$ where $\#(x, M) \neq \mathsf{size}(C)$ and we can perform the reduction:

$$[\texttt{R : Fail}^\ell]\dfrac{\#(x, M) \neq \mathsf{size}(C) \qquad \widetilde{y} = (\mathsf{mlfv}(M) \setminus x) \uplus \mathsf{mlfv}(C)}{M\ \langle\langle C \star U / x \rangle\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \texttt{fail}^{\widetilde{y}}}$$

with $\mathbb{M}' = \sum_{\mathsf{PER}(B)} \texttt{fail}^{\widetilde{y}}$. By hypothesis, one has the derivation:

$$[\texttt{F:ex-sub}]\dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^2 \models M : \tau \qquad \Theta; \Delta \models C \star U : (\sigma^2, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\langle\langle C \star U / x \rangle\rangle : \tau}$$

From $\#(x, M) \neq \mathsf{size}(B)$ we have that $j \neq k$. Hence $\Gamma = \Gamma', \Delta$ and we type the following:

$$[\texttt{F:sum}]\dfrac{[\texttt{F:fail}]\dfrac{}{\Theta; \Gamma \models \texttt{fail}^{\widetilde{y}} : \tau} \quad \cdots \quad [\texttt{F:fail}]\dfrac{}{\Theta; \Gamma \models \texttt{fail}^{\widetilde{y}} : \tau}}{\Theta; \Gamma \models \sum_{\mathsf{PER}(B)} \texttt{fail}^{\widetilde{y}} : \tau}$$

5. **Rule** $[\texttt{R : Fail}^!]$.
Then $\mathbb{M} = M\ \langle\langle C \star U / x \rangle\rangle$ where $\mathsf{head}(M) = x[i]$, $U_i = \mathtt{1}^!$ and we can perform the reduction:

$$[\texttt{R : Fail}^!]\dfrac{\mathsf{head}(M) = x[i] \qquad U_i = \mathtt{1}^!}{M\ \langle\langle C \star U / x \rangle\rangle \longrightarrow M\{\!|\texttt{fail}^\emptyset/x[i]|\!\}\langle\langle C \star U / x \rangle\rangle}$$

with $\mathbb{M}' = M\{\!|\texttt{fail}^\emptyset/x[i]|\!\}\langle\langle C \star U / x \rangle\rangle$. By hypothesis, one has the derivation:

$$[\texttt{F:ex-sub}]\dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau \qquad \Theta; \Delta \models C \star U : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\langle\langle C \star U / x \rangle\rangle : \tau}$$

where $\Gamma = \Gamma', \Delta$. By Lemma 35, there is a derivation Π_1 of $\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!|\texttt{fail}^\emptyset/x[i]|\!\} : \tau$. Therefore, one has the derivation: (the last rule applied is $[\texttt{R : ex-sub}]$)

$$\dfrac{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M\{\!|\mathtt{fail}^{\emptyset}/x[i]|\!\} : \tau \qquad [\text{F:bag}]\ \dfrac{\Theta; \cdot \models U : \epsilon \qquad \Theta;\Delta \models B : \sigma^k}{\Theta;\Delta \models C \star U : (\sigma^k, \epsilon)} \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\{\!|\mathtt{fail}^{\emptyset}/x[i]|\!\}\langle\!\langle C \star U/x \rangle\!\rangle : \tau}$$

6. **Rule** $[\mathtt{R} : \mathtt{Cons}_1]$**.**
Then $\mathbb{M} = \mathtt{fail}^{\widetilde{x}}\ B$ where $B = C \star U$, $C = \wr N_1 \int \cdot \cdots \cdot \wr N_k \int$, $k \geq 0$ and we can perform the following reduction:

$$[\mathtt{R} : \mathtt{Cons}_1]\ \dfrac{\mathsf{size}(C) = k \qquad \widetilde{y} = \mathsf{mlfv}(C)}{\mathtt{fail}^{\widetilde{x}}\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}}}$$

where $\mathbb{M}' = \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}}$. By hypothesis, one has

$$[\text{F:app}]\ \dfrac{[\mathtt{F{:}fail}]\ \dfrac{}{\Theta; \Gamma' \models \mathtt{fail}^{\widetilde{x}} : (\sigma^j, \eta) \to \tau} \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models \mathtt{fail}^{\widetilde{x}}\ B : \tau}$$

Hence $\Gamma = \Gamma', \Delta$ and we may type the following:

$$[\mathtt{F{:}sum}]\ \dfrac{[\mathtt{F{:}fail}]\ \dfrac{}{\Theta; \Gamma \models \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau} \quad \cdots \quad [\mathtt{F{:}fail}]\ \dfrac{}{\Theta; \Gamma \models \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau}}{\Theta; \Gamma \models \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau}$$

7. **Rule** $[\mathtt{R} : \mathtt{Cons}_2]$**.**
Then $\mathbb{M} = \mathtt{fail}^{\widetilde{z}}\ \langle\!\langle B/x \rangle\!\rangle$ where $B = \wr N_1 \int \cdot \cdots \cdot \wr N_k \int$, $k \geq 1$ and one has the reduction:

$$[\mathtt{R} : \mathtt{Cons}_2]\ \dfrac{\#(z, \widetilde{x}) = \mathsf{size}(C) \quad \widetilde{y} = \mathsf{mlfv}(C) \quad \widetilde{y} = \mathsf{mlfv}(C)}{\mathtt{fail}^{\widetilde{x}}\ \langle\!\langle C \star U/z \rangle\!\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{(\widetilde{x} \setminus z) \uplus \widetilde{y}}}$$

where $\mathbb{M}' = \sum_{\mathsf{PER}(B)} \mathtt{fail}^{(\widetilde{z} \setminus x) \uplus \widetilde{y}}$. By hypothesis, there exists a derivation:

$$[\text{F:ex-sub}]\ \dfrac{[\mathtt{F{:}fail}]\ \dfrac{\mathtt{dom}(\Gamma', \hat{x} : \sigma^j) = \widetilde{z}}{\Theta, x^! : \eta; \Gamma', \hat{x} : \sigma^j \models M : \tau} \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models \mathtt{fail}^{\widetilde{z}} \langle\!\langle B/x \rangle\!\rangle : \tau}$$

Hence $\Gamma = \Gamma', \Delta$ and we may type the following:

$$[\mathtt{F{:}sum}]\ \dfrac{[\mathtt{F{:}fail}]\ \dfrac{}{\Theta; \Gamma \models \mathtt{fail}^{(\widetilde{z} \setminus x) \uplus \widetilde{y}} : \tau} \quad \cdots \quad [\mathtt{F{:}fail}]\ \dfrac{}{\Theta; \Gamma \models \mathtt{fail}^{(\widetilde{z} \setminus x) \uplus \widetilde{y}} : \tau}}{\Theta; \Gamma \models \sum_{\mathsf{PER}(B)} \mathtt{fail}^{(\widetilde{z} \setminus x) \uplus \widetilde{y}} : \tau}$$

8. **Rule** $[\mathtt{R} : \mathtt{TCont}]$**.**
Then $\mathbb{M} = C[M]$ and the reduction is as follows:

$$[\mathtt{R} : \mathtt{TCont}]\ \dfrac{M \longrightarrow M_1' + \cdots + M_l'}{C[M] \longrightarrow C[M_1'] + \cdots + C[M_l']}$$

where $\mathbb{M}' = C[M_1'] + \cdots + C[M_l']$. The proof proceeds by analysing the context C:

a. $C = [\cdot]\ B$.
In this case $\mathbb{M} = M\ B$, for some B, and the following derivation holds:

$$[\texttt{F:app}]\ \dfrac{\Theta;\Gamma' \models M : (\sigma^j, \eta) \to \tau \quad \Theta;\Delta \models B : (\sigma^k, \epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models M\ B : \tau}$$

where $\Gamma = \Gamma', \Delta$. Since $\Theta;\Gamma' \models M : (\sigma^j, \eta) \to \tau$ and $M \longrightarrow M'_1 + \ldots + M'_l$, it follows by IH that $\Gamma' \models M'_1 + \ldots + M'_l : (\sigma^j, \eta) \to \tau$. By applying [F:sum], one has $\Theta;\Gamma' \models M'_i : (\sigma^j, \eta) \to \tau$, for $i = 1, \ldots, l$. Therefore, we may type the following:

$$[\texttt{F:sum}]\ \dfrac{\forall i \in 1,\ldots,l \quad [\texttt{F:app}]\ \dfrac{\Theta;\Gamma' \models M'_i : (\sigma^j, \eta) \to \tau \quad \Theta;\Delta \models B : (\sigma^k, \epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models M'_i\ B : \tau}}{\Theta;\Gamma',\Delta \models (M'_1\ B) + \cdots + (M'_l\ B) : \tau}$$

Thus, $\Gamma \models \mathbb{M}' : \tau$, and the result follows.

b. $C = ([\cdot])\langle\langle B/x \rangle\rangle$.
This case is similar to the previous.

9. Rule [R : ECont].
Then $\mathbb{M} = D[\mathbb{M}'']$ where $\mathbb{M}'' \to \mathbb{M}'''$ then we can perform the following reduction:

$$[\texttt{R : ECont}]\ \dfrac{\mathbb{M}'' \longrightarrow \mathbb{M}'''}{D[\mathbb{M}''] \longrightarrow D[\mathbb{M}''']}$$

Hence $\mathbb{M}' = D[\mathbb{M}''']$. The proof proceeds by analysing the context D ($D = [\cdot] + \mathbb{N}$ or $D = \mathbb{N} + [\cdot]$), and follows easily by induction hypothesis.

◀

B.1 Examples

This section contains examples illustrating the constructions and results given in Section 3.

▶ **Example 37.** The following is a wfderivation Π_2 for the bag concatenation $\wr x \int \star 1^!$:

$$[\texttt{F:bag}]\ \dfrac{[\texttt{F:bag}^\ell]\ \dfrac{[\texttt{F:var}^\ell]\ \dfrac{}{\Theta';x:\sigma \models x:\sigma} \quad [\texttt{F:1}^\ell]\ \dfrac{}{\Theta';\text{-} \models 1 : \omega}}{\Theta';x:\sigma \models \wr x \int \cdot 1 : \sigma} \quad [\texttt{F:1}^!]\ \dfrac{}{\Theta';\text{-} \models 1^! : \sigma'}}{\Theta';x:\sigma \models (\wr x \int \star 1^!) : (\sigma, \sigma')}$$

▶ **Example 38** (Cont.37)**.** The following is a well-formedness derivation (labels of the rules being applied are omitted) for term $\Delta_4 = \lambda x.x[1](\wr x \int \star 1^!)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Theta, x^! : (\sigma^j,\eta) \to \tau; x : (\sigma^j,\eta) \to \tau \models x : (\sigma^j,\eta) \to \tau}}{\Theta, x^! : (\sigma^j,\eta) \to \tau; \text{-} \models x[1] : (\sigma^j,\eta) \to \tau} \quad \Pi_2 \quad \eta \propto \sigma'}{\Theta, x^! : (\sigma^j,\eta) \to \tau; x : \sigma \models x[1](\wr x \int \star 1^!) : \tau}}{\Theta;\text{-} \models \lambda x.(x[1](\wr x \int \star 1^!)) : (\sigma, (\sigma^j,\eta) \to \tau) \to \tau}$$

▶ **Example 39.** Below we show the wf-derivation for the bag $A = (\wr x[1] \int \cdot \wr x \int) \star \wr x[2] \int^!$.
First, let Π be the following derivation:

$$[\texttt{F:bag}^\ell]\ \dfrac{[\texttt{F:var}^!]\ \dfrac{[\texttt{F:var}^\ell]\ \dfrac{}{\Theta, x^! : \eta; x : \sigma_3 \models x : \sigma_3}}{\Theta, x^! : \eta; \text{-} \models x[1] : \sigma_3} \quad [\texttt{F:bag}^\ell]\ \dfrac{[\texttt{F:var}^\ell]\ \dfrac{}{\Theta, x^! : \eta; x : \sigma_3 \models x : \sigma_3} \quad [\texttt{F:1}^\ell]\ \dfrac{}{\Theta, x^! : \eta; \text{-} \models 1 : \omega}}{\Theta, x^! : \eta; x : \sigma_3 \models \wr x \int \cdot 1 : \sigma_3}}{\Theta, x^! : \eta; x : \sigma_3 \models \wr x[1] \int \cdot \wr x \int : \sigma_3^2}$$

From Π we can obtain the well-formedness derivation Π_A for A:

$$\dfrac{\dfrac{\Pi}{\Theta, x^! : \eta; x : \sigma_3 \models \wr x[1] \int \cdot \wr x \int : \sigma_3^2} \quad \dfrac{\dfrac{\dfrac{}{\Theta, x^! : \eta; x : \sigma_2 \models x : \sigma_2}\,[\mathtt{F{:}var}^\ell]}{\Theta, x^! : \eta; \text{-} \models x[2] : \sigma_2}\,[\mathtt{F{:}var}^!]}{\Theta, x^! : \eta; \text{-} \models \wr x[2] \int^! : \sigma_2}\,[\mathtt{F{:}bag}^!]}{\Theta, x^! : \eta; x : \sigma_3 \models \underbrace{(\wr x[1] \int \cdot \wr x \int) \star \wr x[2] \int^!}_{A} : (\sigma_3^2, \sigma_2)}\,[\mathtt{F{:}bag}]$$

where $\eta = \sigma_3 \diamond \sigma_2$.

▶ **Example 40.** Below we present the wf-derivation Π_B of the bag $B = \wr x \int \star 1^!$:

$$\dfrac{\dfrac{\dfrac{}{\Theta, x^! : \eta; x : \sigma_3 \models x : \sigma_3}\,[\mathtt{F{:}var}^\ell] \quad \dfrac{}{\Theta, x^! : \eta; \text{-} \models 1 : \omega}\,[\mathtt{F{:}1}^\ell]}{\Theta, x^! : \eta; x : \sigma_3 \models \wr x \int \cdot 1 : \sigma_3^1}\,[\mathtt{F{:}bag}^\ell] \quad \dfrac{}{\Theta, x^! : \eta; \text{-} \models 1^! : \sigma'}\,[\mathtt{F{:}1}^!]}{\Theta, x^! : \eta; x : \sigma_3 \models (\wr x \int \star 1^!) : (\sigma_3, \sigma')}\,[\mathtt{F{:}bag}]$$

▶ **Example 41.** To illustrate our well-formed rules, let M be the following $u\lambda^{\lightning}_{\oplus}$-term:

$$M = \lambda x.(y(\underbrace{(\wr x[1] \int \cdot \wr x \int) \star \wr x[2] \int^!}_{A})\underbrace{(\wr x \int \star 1^!)}_{B}).$$

To ease the notation M is an abstraction $\lambda x.((yA)\ B)$, where $A = (\wr x[1] \int \cdot \wr x \int) \star \wr x[2] \int^!$ and $B = \wr x \int \star 1^!$. From the derivation Π_A (Example 39) we obtain the wf-derivation Π'_A for the application yA:

$$\dfrac{\dfrac{}{\Theta, x^! : \eta; \Delta \models y : (\sigma_3^k, \eta'') \to ((\sigma_3^j, \eta') \to \tau)}\,[\mathtt{F{:}var}^\ell] \quad \dfrac{\Pi_A}{\Theta, x^! : \eta; x : \sigma_3 \models A : (\sigma_3^2, \sigma_2)} \quad \eta'' \propto \sigma_2}{\Theta, x^! : \eta; x : \sigma_3, \Delta \models yA : (\sigma_3^j, \eta') \to \tau}\,[\mathtt{F{:}app}]$$

where $\eta = \sigma_3 \star \sigma_2$, for some list type η' and integers k, j. From the premise $\eta'' \propto \sigma_2$ it follows that $\eta'' = \sigma_2 \diamond \eta'''$ for an arbitrary η'''. From the derivation Π_B (Example 40) we obtain the well-formed derivation for term M:

$$\dfrac{\dfrac{\dfrac{\Pi'_A}{\Theta, x^! : \eta; x : \sigma_3, \Delta \models y\ A : (\sigma_3^j, \eta') \to \tau} \quad \dfrac{\Pi_B}{\Theta, x^! : \eta; x : \sigma_3 \models B : (\sigma_3, \sigma')} \quad \eta' \propto \sigma'}{\Theta, x^! : \eta; x : \sigma_3, x : \sigma_3, \Delta \models (yA)B : \tau}\,[\mathtt{F : app}] \quad x \notin \mathtt{dom}(\Delta)}{\Theta; \Delta \models \lambda x.((yA)B) : (\sigma_3^2, \eta) \to \tau}\,[\mathtt{F{:}abs}]$$

where $\Delta = y : (\sigma_3^k, \eta'') \to ((\sigma_3^j, \eta') \to \tau)$. From the premise $\eta' \propto \sigma'$ we obtain that $\eta' = \sigma' \diamond \eta''''$, where σ' is an arbitrary strict type and η'''' is an arbitrary list type.

C Appendix to Subsection 4.1

▶ **Definition 42** (Structural Congruence). *Structural congruence is defined as the least congruence relation on processes such that:*

$$\begin{array}{ccc}
P \equiv_\alpha Q \Rightarrow P \equiv Q & P \mid \mathbf{0} \equiv P & P \mid Q \equiv Q \mid P \\
(\nu x)\mathbf{0} \equiv \mathbf{0} & (P \mid Q) \mid R \equiv P \mid (Q \mid R) & [x \leftrightarrow y] \equiv [y \leftrightarrow x] \\
\multicolumn{2}{c}{x \notin fn(P) \Rightarrow ((\nu x)P) \mid Q \equiv (\nu x)(P \mid Q)} & (\nu x)(\nu y)P \equiv (\nu y)(\nu x)P \\
\multicolumn{2}{c}{P \oplus (Q \oplus R) \equiv (P \oplus Q) \oplus R} & P \oplus Q \equiv Q \oplus P \\
\multicolumn{2}{c}{(\nu x)(P \mid (Q \oplus R)) \equiv (\nu x)(P \mid Q) \oplus (\nu x)(P \mid R)} & \mathbf{0} \oplus \mathbf{0} \equiv \mathbf{0}
\end{array}$$

D Appendix to Subsection 4.2

We need a few auxiliary notions to formalize reduction for $u\widehat{\lambda}^{\lightning}_{\oplus}$.

▶ **Definition 43** (Head). *We amend Definition 6 for the case of terms in* $u\widehat{\lambda}^{\lightning}_{\oplus}$*:*

$$\mathsf{head}(x) = x \qquad \mathsf{head}(x[i]) = x[i]$$
$$\mathsf{head}(M\ B) = \mathsf{head}(M) \qquad \mathsf{head}(\lambda x.(M[\widetilde{x} \leftarrow x])) = \lambda x.(M[\widetilde{x} \leftarrow x])$$
$$\mathsf{head}(M\langle\!| N/x |\!\rangle) = \mathsf{head}(M) \qquad \mathsf{head}(M\llbracket U/x \rrbracket) = \mathsf{head}(M)$$
$$\mathsf{head}((M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle) = (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle \qquad \mathsf{head}(\mathtt{fail}^{\widetilde{x}}) = \mathtt{fail}^{\widetilde{x}}$$
$$\mathsf{head}(M[\widetilde{x} \leftarrow x]) = \begin{cases} x & \text{If } \mathsf{head}(M) = y \text{ and } y \in \widetilde{x} \\ \mathsf{head}(M) & \text{Otherwise} \end{cases}$$

▶ **Definition 44** (Linear Head Substitution). *Given an M with* $\mathsf{head}(M) = x$*, the linear substitution of a term N for the head variable x of the term M, written $M\{\!| N/x |\!\}$ is inductively defined as:*

$$x\{\!| N/x |\!\} = N \qquad (M\ B)\{\!| N/x |\!\} = (M\{\!| N/x |\!\})\ B$$
$$(M\llbracket U/y \rrbracket)\{\!| N/x |\!\} = (M\{\!| N/x |\!\})\ \llbracket U/y \rrbracket \qquad x \neq y$$
$$(M\langle\!| L/y |\!\rangle)\{\!| N/x |\!\} = (M\{\!| N/x |\!\})\ \langle\!| L/y |\!\rangle \qquad x \neq y$$
$$((M[\widetilde{y} \leftarrow y])\langle\!\langle B/y \rangle\!\rangle)\{\!| N/x |\!\} = (M[\widetilde{y} \leftarrow y]\{\!| N/x |\!\})\ \langle\!\langle B/y \rangle\!\rangle \qquad x \neq y$$
$$(M[\widetilde{y} \leftarrow y])\{\!| N/x |\!\} = (M\{\!| N/x |\!\})[\widetilde{y} \leftarrow y] \qquad x \neq y$$

Following Def. 8, we define contexts for terms and expressions. While expression contexts are as in Def. 8; the term contexts for $u\widehat{\lambda}^{\lightning}_{\oplus}$ involve explicit linear and unrestricted substitutions, rather than an explicit substitution: this is due to the reduction strategy we have chosen to adopt, as we always wish to evaluate explicit substitutions first. We assume that the terms that fill in the holes respect the conditions on explicit linear substitutions (i.e., variables appear in a term only once, shared variables must occur in the context), similarly for explicit unrestricted substitutions.

▶ **Definition 45** (Evaluation Contexts). *Contexts for terms and expressions are defined by the following grammar:*

$$\begin{aligned} C[\cdot], C'[\cdot] &::= ([\cdot])B \mid ([\cdot])\langle\!| N/x |\!\rangle \mid ([\cdot])\llbracket U/x \rrbracket \mid ([\cdot])[\widetilde{x} \leftarrow x] \mid ([\cdot])[\leftarrow x]\langle\!\langle 1/x \rangle\!\rangle \\ D[\cdot], D'[\cdot] &::= M + [\cdot] \mid [\cdot] + M \end{aligned}$$

The result of replacing a hole with a $u\widehat{\lambda}^{\lightning}_{\oplus}$-term M in a context $C[\cdot]$, denoted with $C[M]$, has to be a term in $u\widehat{\lambda}^{\lightning}_{\oplus}$.

This way, e.g., the hole in context $C[\cdot] = ([\cdot])\langle\!| N/x |\!\rangle$ cannot be filled with y, since $C[y] = (y)\langle\!| N/x |\!\rangle$ is not a well-defined term. Indeed, $M\langle\!| N/x |\!\rangle$ requires that x occurs exactly once within M. Similarly, we cannot fill the hole with $\mathtt{fail}^{z}$ with $z \neq x$, since $C[\mathtt{fail}^{z}] = (\mathtt{fail}^{z})\langle\!| N/x |\!\rangle$ is also not a well-defined term, for the same reason.

Operational Semantics

As in $u\lambda^{\lightning}_{\oplus}$, the reduction relation $\longrightarrow$ on $u\widehat{\lambda}^{\lightning}_{\oplus}$ operates lazily on expressions; it is defined by the rules in Fig. 13, and relies on a notion of linear free variables given in Fig. 12.

As expected, rule $[\mathtt{RS} : \mathtt{Beta}]$ results into an explicit substitution $M[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle$, where $B = C \star U$ is a bag with a linear part C and an unrestricted part U.

$$\begin{array}{ll}
\mathsf{lfv}(x) = \{x\} & \mathsf{lfv}(M\ B) = \mathsf{lfv}(M) \cup \mathsf{lfv}(B) \\
\mathsf{lfv}(x[i]) = \emptyset & \mathsf{lfv}(\lambda x.M[\widetilde{x} \leftarrow x]) = \mathsf{lfv}(M[\widetilde{x} \leftarrow x]) \setminus \{x\} \\
\mathsf{lfv}(1) = \emptyset & \mathsf{lfv}(M[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle) = (\mathsf{lfv}(M[\widetilde{x} \leftarrow x]) \setminus \{x\}) \uplus \mathsf{lfv}(B) \\
\mathsf{lfv}(\wr M \int) = \mathsf{lfv}(M) & \mathsf{lfv}(M \langle\!| N/x |\!\rangle) = \mathsf{lfv}(M) \cup \mathsf{lfv}(N) \\
\mathsf{lfv}(\wr M \int^!) = \mathsf{lfv}(M) & \mathsf{lfv}(M \llbracket U/x \rrbracket) = \mathsf{lfv}(M) \\
\mathsf{lfv}(C \star U) = \mathsf{lfv}(C) & \mathsf{lfv}(\mathbb{M} + \mathbb{N}) = \mathsf{lfv}(\mathbb{M}) \cup \mathsf{lfv}(\mathbb{N}) \\
\mathsf{lfv}(\wr M \int \cdot C) = \mathsf{lfv}(M) \cup \mathsf{lfv}(C) & \mathsf{lfv}(\mathtt{fail}^{x_1, \cdots, x_n}) = \{x_1, \ldots, x_n\}
\end{array}$$

Figure 12 Free Variables for $u\widehat{\lambda}^{\lightning}_{\oplus}$.

$$[\mathsf{RS{:}Beta}]\ \frac{}{(\lambda x.(M[\widetilde{x} \leftarrow x]))B \longrightarrow (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle}$$

$$[\mathtt{RS{:}Fetch}^{\ell}]\ \frac{\mathsf{head}(M) = x}{M \langle\!| N/x |\!\rangle \longrightarrow M\{\!| N/x |\!\}} \qquad [\mathtt{RS{:}Fetch}^{!}]\ \frac{\mathsf{head}(M) = x[i] \quad U_i = \wr N \int^!}{M \llbracket U/x \rrbracket \longrightarrow M\{\!| N/x[i] |\!\}\llbracket U/x \rrbracket}$$

$$[\mathsf{RS{:}Ex\text{-}Sub}]\ \frac{C = \wr M_1 \int \cdots \wr M_k \int \qquad M \neq \mathtt{fail}^{\widetilde{y}}}{M[x_1, \cdots, x_k \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \sum_{C_i \in \mathsf{PER}(C)} M \langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket}$$

$$[\mathtt{RS{:}Fail}^{\ell}]\ \frac{k \neq \mathsf{size}(C) \qquad \widetilde{y} = (\mathsf{lfv}(M) \setminus \{\widetilde{x}\}) \cup \mathsf{lfv}(C)}{M[x_1, \cdots, x_k \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \sum_{C_i \in \mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}}}$$

$$[\mathtt{RS{:}Fail}^{!}]\ \frac{U_i = 1^! \quad \mathsf{head}(M) = x[i] \qquad \widetilde{y} = \mathsf{lfv}(M)}{M \llbracket U/x \rrbracket \longrightarrow M\{\!| \mathtt{fail}^{\emptyset}/x[i] |\!\}\llbracket U/x \rrbracket}$$

$$[\mathtt{RS{:}Cons_1}]\ \frac{\widetilde{y} = \mathsf{lfv}(C)}{\mathtt{fail}^{\widetilde{x}}\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}}}$$

$$[\mathtt{RS{:}Cons_2}]\ \frac{\mathsf{size}(C) = |\widetilde{x}| \qquad \widetilde{z} = \mathsf{lfv}(C)}{(\mathtt{fail}^{\widetilde{x} \uplus \widetilde{y}}[\widetilde{x} \leftarrow x])\langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y} \uplus \widetilde{z}}}$$

$$[\mathtt{RS{:}Cons_3}]\ \frac{\widetilde{z} = \mathsf{lfv}(N)}{\mathtt{fail}^{\widetilde{y} \cup x} \langle\!| N/x |\!\rangle \longrightarrow \mathtt{fail}^{\widetilde{y} \cup \widetilde{z}}} \qquad [\mathtt{RS{:}Cons_4}]\ \frac{}{\mathtt{fail}^{\widetilde{y}} \llbracket U/x \rrbracket \longrightarrow \mathtt{fail}^{\widetilde{y}}}$$

Figure 13 Reduction Rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$ (contextual rules omitted)

In the case $|\widetilde{x}| = k = \mathsf{size}(C)$ and $M \neq \mathtt{fail}^{\widetilde{y}}$, this explicit substitution expands into a sum of terms involving explicit linear and unrestricted substitutions $\langle\!| N/x |\!\rangle$ and $\llbracket U/x \rrbracket$, which are the ones to reduce into a head substitution, via rule [RS:Ex-Sub]. Intuitively, rule [RS:Ex-Sub] "distributes" an explicit substitution into a sum of terms involving explicit linear substitutions; it considers all possible permutations of the elements in the bag among all shared variables. Explicit linear/unrestricted substitutions evolve either into a head substitution $\{\!|N/x|\!\}$ (with $N \in B$), via rule $[\mathtt{RS:Fetch}^{\ell}]$, or $\{\!|N/x|\!\}\llbracket U/x \rrbracket$ (with $U \in B$) via rule $[\mathtt{RS:Fetch}^{!}]$, depending on whether the head of the term is a linear or an unrestricted variable.

In the case $|\widetilde{x}| = k \neq \mathsf{size}(C)$ or $M = \mathtt{fail}^{\widetilde{y}}$, the term $M[\widetilde{x} \leftarrow x]\langle\!\langle B/x \rangle\!\rangle$ will be a redex of either rule $[\mathtt{RS:Fail}^{\ell}]$ or $[\mathtt{RS:Cons_2}]$. The latter has a side condition $|\widetilde{x}| = \mathsf{size}(C)$, because we want to give priority for application of $[\mathtt{RS:Fail}^{\ell}]$ when there is a mismatch of linear variables and the number of linear resources. Rule $[\mathtt{RS:Fail}^{!}]$ applies to an unrestricted substitution $M\llbracket U/x \rrbracket$ when the head of M is an unrestricted variable, say $x[i]$, that aims to consume the i-th component of the bag U which is empty, i.e., $U_i = 1^!$; then the term reduces to a term where all the head of M is substituted by $\mathtt{fail}^{\emptyset}$, the explicit unrestricted substitution is not consumed and continues in the resulting term. Consuming rules $[\mathtt{RS:Cons_1}], [\mathtt{RS:Cons_3}]$ and $[\mathtt{RS:Cons_4}]$ the term $\mathtt{fail}$ consume either a bag, or an explicit linear substitution, or an explicit unrestricted substitution, respectively.

Notice that the left-hand sides of the reduction rules in $u\widehat{\lambda}^{\lightning}_{\oplus}$ do not interfere with each other. Similarly to $u\lambda^{\lightning}_{\oplus}$, reduction in $u\widehat{\lambda}^{\lightning}_{\oplus}$ satisfies a *diamond property.*

▶ **Example 46.** We continue to illustrate the different behaviors of the terms below w.r.t. the reduction rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$ (Fig. 13):

1. The case with a linear variable x in which the linear bag has size one, is close to the standard *meaning* of applying an identity function to a term:

$$(\lambda x.x_1[x_1 \leftarrow x])\wr N' \int \star U' \longrightarrow_{[\mathtt{R:Beta}]} x_1[x_1 \leftarrow x]\langle\!\langle \wr N' \int \star U'/x \rangle\!\rangle$$
$$\longrightarrow_{[\mathtt{RS:Ex\text{-}Sub}]} x_1\langle\!| N'/x_1 |\!\rangle \llbracket U'/x \rrbracket \longrightarrow_{[\mathtt{R:Fetch}^{\ell}]} x_1\{\!|N'/x_1|\!\}\llbracket U'/x \rrbracket = N'\llbracket U'/x \rrbracket$$

2. The case of an abstraction of one unrestricted variable that aims to consume the first element of the unrestricted bag, which fails to contain a resource in the first component.

$$(\lambda x.x[1][\leftarrow x])\mathtt{1} \star \mathtt{1}^! \diamond U' \longrightarrow_{[\mathtt{R:Beta}]} x[1][\leftarrow x]\langle\!\langle \mathtt{1} \star \mathtt{1}^! \diamond U'/x \rangle\!\rangle \longrightarrow_{[\mathtt{RS:Ex\text{-}Sub}]} x[1]\llbracket \mathtt{1}^! \diamond U'/x \rrbracket$$
$$\longrightarrow_{[\mathtt{RS:Fail}^!]} x[1]\{\!|\mathtt{fail}^{\emptyset}/x[1]|\!\}\llbracket \mathtt{1}^! \diamond U'/x \rrbracket = \mathtt{fail}^{\emptyset}\llbracket \mathtt{1}^! \diamond U'/x \rrbracket$$

3. The case of an abstraction of one unrestricted variable that aims to consume the ith component of the unrestricted bag U'. In the case $C' = \mathtt{1}$ and $U'_i \neq 1^!$ the reduction is:

$$(\lambda x.x[i][\leftarrow x])C' \diamond U' \longrightarrow_{[\mathtt{R:Beta}]} x[i][\leftarrow x]\langle\!\langle C' \diamond U'/x \rangle\!\rangle$$
$$\longrightarrow_{[\mathtt{RS:Ex\text{-}Sub}]} x[i]\llbracket C' \diamond U'/x \rrbracket$$
$$\longrightarrow_{[\mathtt{RS:Fetch}^!]} x[i]\{\!|N'/x[i]|\!\}\llbracket C' \diamond U'/x \rrbracket = N\llbracket C' \diamond U'/x \rrbracket$$

where $U'_i = \wr N' \int^!$. Otherwise, $U'_i = 1^!$ and the reduction relies again on the size of the linear bag C: if $\#(x, x[i]) = \mathsf{size}(C')$ the reduction ends with an application of $[\mathtt{R:fail}^!]$; otherwise, it ends with an application $[\mathtt{R:fail}^{\ell}]$.

D.1 Well-formedness rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$

Similarly to $u\lambda^{\lightning}_{\oplus}$ we present a set "well-formedness" rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$-terms, -bags and -expressions, based on an intersection type system for $u\widehat{\lambda}^{\lightning}_{\oplus}$, defined upon strict, multiset, list, tuple types, as introduced for $u\lambda^{\lightning}_{\oplus}$ and presented in Fig. 14. Linear contexts Γ, Δ and unrestricted contexts Θ, Υ are the same as in $u\lambda^{\lightning}_{\oplus}$, as well as well-formedness judgements $\Theta; \Gamma \vdash \mathbb{M} : \sigma$.

$$[\texttt{FS:var}^{\ell}]\ \frac{}{\Theta; x:\sigma \models x:\sigma} \qquad [\texttt{FS:var}^{!}]\ \frac{\Theta, x:\eta; x:\eta_i, \Delta \models x:\sigma}{\Theta, x:\eta; \Delta \models x[i]:\sigma} \qquad [\texttt{FS:1}^{\ell}]\ \frac{}{\Theta; \text{-} \models 1:\omega}$$

$$[\texttt{FS:1}^{!}]\ \frac{}{\Theta; \text{-} \models 1^{!}:\sigma} \qquad [\texttt{FS:weak}]\ \frac{\Theta;\Gamma \models M:\tau}{\Theta;\Gamma, x:\omega \models M[\leftarrow x]:\tau}$$

$$[\texttt{FS:sum}]\ \frac{\Theta;\Gamma \models \mathbb{M}:\sigma \quad \Theta;\Gamma \models \mathbb{N}:\sigma}{\Theta;\Gamma \models \mathbb{M}+\mathbb{N}:\sigma} \qquad [\texttt{FS:abs-sh}]\ \frac{\Theta, x:\eta;\Gamma, x:\sigma^k \models M[\widetilde{x} \leftarrow x]:\tau \quad x \notin \texttt{dom}(\Gamma)}{\Theta;\Gamma \models \lambda x.(M[\widetilde{x} \leftarrow x]):(\sigma^k,\eta) \to \tau}$$

$$[\texttt{FS:app}]\ \frac{\Theta;\Gamma \models M:(\sigma^j,\eta)\to\tau \quad \Theta;\Delta \models B:(\sigma^k,\epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma,\Delta \models M\ B:\tau}$$

$$[\texttt{FS:bag}]\ \frac{\Theta;\Gamma \models C:\sigma^k \quad \Theta;\text{-} \models U:\eta}{\Theta;\Gamma \models C \star U:(\sigma^k,\eta)} \qquad [\texttt{FS:}\diamond-\texttt{bag}^{!}]\ \frac{\Theta;\text{-} \models U:\epsilon \quad \Theta;\text{-} \models V:\eta}{\Theta;\text{-} \models U \diamond V:\epsilon \diamond \eta}$$

$$[\texttt{FS:bag}^{!}]\ \frac{\Theta;\text{-} \models M:\sigma}{\Theta;\text{-} \models \wr M \int^{!}:\sigma} \qquad [\texttt{FS:bag}^{\ell}]\ \frac{\Theta;\Gamma \models M:\sigma \quad \Theta;\Delta \models C:\sigma^k}{\Theta;\Gamma,\Delta \models \wr M \int \cdot C:\sigma^{k+1}}$$

$$[\texttt{FS:share}]\ \frac{x \notin \texttt{dom}(\Gamma) \quad k \neq 0 \quad \Theta;\Gamma, x_1:\sigma,\cdots,x_k:\sigma \models M:\tau}{\Theta;\Gamma, x:\sigma^k \models M[x_1,\cdots,x_k \leftarrow x]:\tau} \qquad [\texttt{FS:Esub}^{\ell}]\ \frac{\Theta;\Gamma, x:\sigma \models M:\tau \quad \Theta;\Delta \models N:\sigma}{\Theta;\Gamma,\Delta \models M\langle\!| N/x |\!\rangle:\tau}$$

$$[\texttt{FS:Esub}^{!}]\ \frac{\Theta, x:\eta;\Gamma \models M:\tau \quad \Theta;\text{-} \models U:\epsilon \quad \eta \propto \epsilon}{\Theta;\Gamma \models M[\![U/x]\!]:\tau}$$

$$[\texttt{FS:Esub}]\ \frac{\Theta;\Delta \models B:(\sigma^k,\epsilon) \quad \Theta, x:\eta;\Gamma, x:\sigma^j \models M[\widetilde{x} \leftarrow x]:\tau \quad \eta \propto \epsilon}{\Theta;\Gamma,\Delta \models (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle:\tau} \qquad [\texttt{FS:fail}]\ \frac{\texttt{dom}(\Gamma) = \widetilde{x}}{\Theta;\Gamma \models \texttt{fail}^{\widetilde{x}}:\tau}$$

Figure 14 Well-Formedness Rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$

▶ **Definition 47** (Well-formedness in $u\widehat{\lambda}^{\lightning}_{\oplus}$). *An expression* $\mathbb{M}$ *is well formed if there exists a* Θ,Γ *and a* τ *such that* $\Theta;\Gamma \models \mathbb{M}:\tau$ *is entailed via the rules in Fig. 14.*

Well-formed rules for $u\widehat{\lambda}^{\lightning}_{\oplus}$ are essentially the same as the ones for $u\lambda^{\lightning}_{\oplus}$. Rules [FS : abs-sh] and [FS:Esub] are modified to take into account the sharing construct $[\widetilde{x} \leftarrow x]$. Rule [FS:share] is exclusive for $u\widehat{\lambda}^{\lightning}_{\oplus}$ and requires, for each $i = 1,\dots,k$, the variable assignment $x_i:\sigma$, to derive the well-formedness of $M[x_1,\dots,x_n \leftarrow x]:\tau$ (in addition to variable assignments in Θ and Γ).

▶ **Lemma 48** (Linear Substitution Lemma for $u\widehat{\lambda}^{\lightning}_{\oplus}$). *If* $\Theta;\Gamma, x:\sigma \models M:\tau$, $\mathsf{head}(M) = x$, *and* $\Theta;\Delta \models N:\sigma$ *then* $\Gamma,\Delta \models M\{\!| N/x |\!\}:\tau$.

Proof. By structural induction on M with $\mathsf{head}(M) = x$. There are six cases to be analyzed:

1. $M = x$

   In this case, $\Theta; x:\sigma \models x:\sigma$ and $\Gamma = \emptyset$. Observe that $x\{\!| N/x |\!\} = N$, since $\Delta \models N:\sigma$, by hypothesis, the result follows.

2. $M = M'\ B$.
Then $\mathsf{head}(M'\ B) = \mathsf{head}(M') = x$, and the derivation is the following:

$$[\texttt{FS:app}]\ \frac{\Theta; \Gamma_1, x : \sigma \models M' : (\delta^j, \eta) \to \tau \quad \Theta; \Gamma_2 \models B : (\delta^k, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, x : \sigma \models M'B : \tau}$$

where $\Gamma = \Gamma_1, \Gamma_2$, and j, k are non-negative integers, possibly different. Since $\Delta \vdash N : \sigma$, by IH, the result holds for M', that is,

$$\Gamma_1, \Delta \models M'\{\!|N/x|\!\} : (\delta^j, \eta) \to \tau$$

which gives the derivation:

$$[\texttt{FS:app}]\ \frac{\Theta; \Gamma_1, \Delta \models M'\{\!|N/x|\!\} : (\delta^j, \eta) \to \tau \quad \Theta; \Gamma_2 \models B : (\delta^k, \epsilon) \quad \eta \propto \epsilon}{\Theta; \Gamma_1, \Gamma_2, \Delta \models (M'\{\!|N/x|\!\})B : \tau}$$

From Def. 44, $(M'B)\{\!|N/x|\!\} = (M'\{\!|N/x|\!\})B$, and the result follows.

3. $M = M'[\widetilde{y} \leftarrow y]$.
Then $\mathsf{head}(M'[\widetilde{y} \leftarrow y]) = \mathsf{head}(M') = x$, for $y \neq x$. Therefore,

$$[\texttt{FS:share}]\ \frac{\Theta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta, x : \sigma \models M' : \tau \quad y \notin \Gamma_1 \quad k \neq 0}{\Theta; \Gamma_1, y : \delta^k, x : \sigma \models M'[y_1, \cdots, y_k \leftarrow x] : \tau}$$

where $\Gamma = \Gamma_1, y : \delta^k$. By IH, the result follows for M', that is,

$$\Theta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta, \Delta \models M'\{\!|N/x|\!\} : \tau$$

and we have the derivation:

$$[\texttt{FS:share}]\ \frac{\Theta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta, \Delta \models M'\{\!|N/x|\!\} : \tau \quad y \notin \Gamma_1 \quad k \neq 0}{\Theta; \Gamma_1, y : \delta^k, \Delta \models M'\{\!|N/x|\!\}[\widetilde{y} \leftarrow y] : \tau}$$

From Def. 44 $M'[\widetilde{y} \leftarrow y]\{\!|N/x|\!\} = M'\{\!|N/x|\!\}[\widetilde{y} \leftarrow y]$, and the result follows.

4. $M = M'[\leftarrow y]$.
Then $\mathsf{head}(M'[\leftarrow y]) = \mathsf{head}(M') = x$ with $x \neq y$,

$$[\texttt{FS:weak}]\ \frac{\Theta; \Gamma, x : \sigma \models M : \tau}{\Theta; \Gamma, y : \omega, x : \sigma \models M[\leftarrow y] : \tau}$$

and $M'[\leftarrow y]\{\!|N/x|\!\} = M'\{\!|N/x|\!\}[\leftarrow y]$. Then by the induction hypothesis:

$$[\texttt{FS:weak}]\ \frac{\Theta; \Gamma, \Delta \models M\{\!|N/x|\!\} : \tau}{\Theta; \Gamma, y : \omega, \Delta \models M\{\!|N/x|\!\}[\leftarrow y] : \tau}$$

5. If $M = M'\langle\!|M''/y|\!\rangle$ then $\mathsf{head}(M'\langle\!|M''/y|\!\rangle) = \mathsf{head}(M') = x \neq y$,

$$[\texttt{FS:ex-sub}^{\ell}]\ \frac{\Theta; \Gamma, y : \delta, x : \sigma \models M : \tau \quad \Theta; \Delta \models M'' : \delta}{\Theta; \Gamma_1, \Gamma_2, x : \sigma \models M'\langle\!|M''/y|\!\rangle : \tau}$$

and $M'\langle\!|M''/y|\!\rangle\{\!|N/x|\!\} = M'\{\!|N/x|\!\}\langle\!|M''/y|\!\rangle$. Then by the induction hypothesis:

$$[\texttt{FS:ex-sub}^{\ell}]\ \frac{\Theta; \Gamma, y : \delta, \Delta \models M'\{\!|N/x|\!\} : \tau \quad \Theta; \Delta \models M'' : \delta}{\Theta; \Gamma_1, \Gamma_2, \Delta \models M'\{\!|N/x|\!\}\langle\!|M''/y|\!\rangle : \tau}$$

6. If $M = M'[\![U/y]\!]$ then $\mathsf{head}(M'[\![U/y]\!]) = \mathsf{head}(M') = x$, and the proofs is similar to the case above.

◀

▶ **Lemma 49** (Unrestricted Substitution Lemma for $u\widehat{\lambda}^{\lightning}_{\oplus}$). *If* $\Theta, x^! : \eta; \Gamma \models M : \tau$, $\mathsf{head}(M) = x[i]$, $\eta_i = \sigma$, *and* $\Theta; \cdot \models N : \sigma$ *then* $\Theta, x^! : \eta; \Gamma \models M\{\!|N/x[i]|\!\}$.

Proof. By structural induction on M with $\mathsf{head}(M) = x[i]$. There are three cases to be analyzed:

1. $M = x[i]$.
   In this case,
$$\dfrac{[\texttt{F:var}^{!}]\quad \dfrac{[\texttt{F:var}^{\ell}]\quad}{\Theta, x^! : \eta; x : \eta_i \models x : \sigma}}{\Theta, x^! : \eta; \cdot \models x[i] : \sigma}$$
   and $\Gamma = \emptyset$. Observe that $x[i]\{\!|N/x[i]|\!\} = N$, since $\Theta, x^! : \eta; \Gamma \models M\{\!|N/x[i]|\!\}$, by hypothesis, the result follows.
2. $M = M'\ B$.
   In this case, $\mathsf{head}(M'\ B) = \mathsf{head}(M') = x[i]$, and one has the following derivation:
$$[\texttt{F:app}]\ \dfrac{\Theta, x^! : \eta; \Gamma_1 \models M : (\delta^j, \epsilon) \to \tau \quad \Theta, x^! : \sigma; \Gamma_2 \models B : (\delta^k, \epsilon') \qquad \epsilon \propto \epsilon'}{\Theta, x^! : \eta; \Gamma_1, \Gamma_2 \models M\ B : \tau}$$
   where $\Gamma = \Gamma_1, \Gamma_2$, δ is a strict type and j, k are non-negative integers, possibly different. By IH, we get $\Theta, x^! : \eta; \Gamma_1 \models M'\{\!|N/x[i]|\!\} : (\delta^j, \epsilon) \to \tau$, which gives the derivation:
$$[\texttt{F:app}]\ \dfrac{\Theta, x^! : \eta; \Gamma_1 \models M'\{\!|N/x[i]|\!\} : (\delta^j, \epsilon) \to \tau \quad \Theta, x^! : \eta; \Gamma_2 \models B : (\delta^k, \epsilon') \qquad \epsilon \propto \epsilon'}{\Theta, x^! : \eta; \Gamma_1, \Gamma_2 \models (M'\{\!|N/x[i]|\!\})B : \tau}$$
   From Def. 7, $M'\langle\!\langle B/y \rangle\!\rangle\{\!|N/x[i]|\!\} = M'\{\!|N/x[i]|\!\}\langle\!\langle B/y \rangle\!\rangle$, and the result follows.
3. $M = M'[\widetilde{y} \leftarrow y]$.
   Then $\mathsf{head}(M'[\widetilde{y} \leftarrow y]) = \mathsf{head}(M') = x[i]$, for $y \neq x$. Therefore,
$$[\texttt{FS:share}]\ \dfrac{\Theta, x^! : \eta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta \models M' : \tau \quad y \notin \Gamma_1 \quad k \neq 0}{\Theta, x^! : \eta; \Gamma_1, y : \delta^k \models M'[y_1, \cdots, y_k \leftarrow y] : \tau}$$
   where $\Gamma = \Gamma_1, y : \delta^k$. By IH, the result follows for M', that is,
$$\Theta, x^! : \eta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta \models M'\{\!|N/x[i]|\!\} : \tau$$
   and we have the derivation:
$$[\texttt{FS:share}]\ \dfrac{\Theta, x^! : \eta; \Gamma_1, y_1 : \delta, \cdots, y_k : \delta \models M'\{\!|N/x[i]|\!\} : \tau \quad y \notin \Gamma_1 \quad k \neq 0}{\Theta, x^! : \eta; \Gamma_1, y : \delta^k \models M'\{\!|N/x[i]|\!\}[\widetilde{y} \leftarrow y] : \tau}$$
   From Def. 44 $M'[\widetilde{y} \leftarrow y]\{\!|N/x[i]|\!\} = M'\{\!|N/x[i]|\!\}[\widetilde{y} \leftarrow y]$, and the result follows.
4. $M = M'[\leftarrow y]$.
   Then $\mathsf{head}(M'[\leftarrow y]) = \mathsf{head}(M') = x[i]$ with $x \neq y$,
$$[\texttt{FS:weak}]\ \dfrac{\Theta, x^! : \eta; \Gamma \models M : \tau}{\Theta, x^! : \eta; \Gamma, y : \omega \models M[\leftarrow y] : \tau}$$
   and $M'[\leftarrow y]\{\!|N/x[i]|\!\} = M'\{\!|N/x[i]|\!\}[\leftarrow y]$. Then by the induction hypothesis:
$$[\texttt{FS:weak}]\ \dfrac{\Theta, x^! : \eta; \Gamma \models M\{\!|N/x[i]|\!\} : \tau}{\Theta, x^! : \eta; \Gamma, y : \omega \models M\{\!|N/x[i]|\!\}[\leftarrow y] : \tau}$$
5. $M = M'\langle\!| M''/y |\!\rangle$.
   Then $\mathsf{head}(M'\langle\!| M''/y |\!\rangle) = \mathsf{head}(M') = x[i]$ with $x \neq y$,

$$[\texttt{FS:ex-sub}^\ell]\ \frac{\Theta, x^! : \eta; \Gamma, y : \delta \models M : \tau \qquad \Theta, x^! : \eta; \Delta \models M'' : \delta}{\Theta, x^! : \eta; \Gamma, \Delta \models M' \langle\!| M''/y |\!\rangle : \tau}$$

and $M'\langle\!| M''/y |\!\rangle \{\!| N/x[i] |\!\} = M'\{\!| N/x[i] |\!\}\langle\!| M''/y |\!\rangle$. Then by the induction hypothesis:

$$[\texttt{FS:ex-sub}^\ell]\ \frac{\Theta, x^! : \eta; \Gamma, y : \delta \models M'\{\!| N/x[i] |\!\} : \tau \qquad \Theta, x^! : \eta; \Delta \models M'' : \delta}{\Theta; \Gamma, \Delta \models M'\{\!| N/x[i] |\!\}\langle\!| M''/y |\!\rangle : \tau}$$

6. $M = M'\llbracket U/y \rrbracket$.
Then $\mathsf{head}(M'\llbracket U/y \rrbracket) = \mathsf{head}(M') = x[i]$,

$$[\texttt{FS:ex-sub}^!]\ \frac{\Theta, x^! : \eta, y^! : \epsilon; \Gamma \models M : \tau \quad \Theta, x^! : \eta; \text{-} \models U : \epsilon}{\Theta, x^! : \eta; \Gamma \models M\llbracket U/y \rrbracket : \tau}$$

and $M'\llbracket U/y \rrbracket\{\!| N/x[i] |\!\} = M'\{\!| N/x[i] |\!\}\llbracket U/y \rrbracket$. Then by the induction hypothesis:

$$[\texttt{FS:ex-sub}^!]\ \frac{\Theta, x^! : \eta, y^! : \epsilon; \Gamma \models M'\{\!| N/x[i] |\!\} : \tau \quad \Theta, x^! : \eta; \text{-} \models U : \eta}{\Theta, x^! : \eta; \Gamma \models M'\{\!| N/x[i] |\!\}\llbracket U/y \rrbracket : \tau}$$

◀

▶ **Theorem 50** (SR in $u\widehat{\lambda}^{\lightning}_{\oplus}$). *If* $\Theta; \Gamma \models \mathbb{M} : \tau$ *and* $\mathbb{M} \longrightarrow \mathbb{M}'$ *then* $\Theta; \Gamma \models \mathbb{M}' : \tau$.

Proof. By structural induction on the reduction rule from Fig. 13 applied in $\mathbb{M} \longrightarrow \mathbb{N}$.

1. Rule [RS:Beta].
Then $\mathbb{M} = (\lambda x.M[\widetilde{x} \leftarrow x])B$ and the reduction is:

$$[\texttt{RS:Beta}]\ \frac{}{(\lambda x.M[\widetilde{x} \leftarrow x])B \longrightarrow M[\widetilde{x} \leftarrow x]\ \langle\!\langle B/x \rangle\!\rangle}$$

where $\mathbb{M}' = M[\widetilde{x} \leftarrow x]\ \langle\!\langle B/x \rangle\!\rangle$. Since $\Theta; \Gamma \models \mathbb{M} : \tau$ we get the following derivation:

$$\begin{array}{l} [\texttt{FS:app}]\ \dfrac{[\texttt{FS:abs-sh}]\ \dfrac{[\texttt{FS:share}]\ \dfrac{\Theta, x^! : \eta; \Gamma', x_1 : \sigma, \cdots, x_j : \sigma \models M : \tau}{\Theta, x^! : \eta; \Gamma', x : \sigma^j \models M[\widetilde{x} \leftarrow x] : \tau}}{\Theta; \Gamma' \models \lambda x.M[\widetilde{x} \leftarrow x] : (\sigma^j, \eta) \to \tau} \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models (\lambda x.M[\widetilde{x} \leftarrow x])B : \tau} \end{array}$$

for $\Gamma = \Gamma', \Delta$ and $x \notin \texttt{dom}(\Gamma')$. Notice that:

$$[\texttt{FS:ex-sub}]\ \dfrac{[\texttt{FS:share}]\ \dfrac{\Theta, x^! : \eta; \Gamma', x_1 : \sigma, \cdots, x_j : \sigma \models M : \tau}{\Theta, x^! : \eta; \Gamma', x : \sigma^j \models M[\widetilde{x} \leftarrow x] : \tau} \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M[\widetilde{x} \leftarrow x]\ \langle\!\langle B/x \rangle\!\rangle : \tau}$$

Therefore $\Theta; \Gamma', \Delta \models \mathbb{M}' : \tau$ and the result follows.

2. Rule [RS:Ex-Sub].
Then $\mathbb{M} = M[x_1, \cdots, x_k \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle$ where $C = \wr N_1 \int \cdot \cdots \cdot \wr N_k \int$. The reduction is:

$$[\texttt{RS:Ex-Sub}]\ \frac{C = \wr M_1 \int \cdots \wr M_k \int \qquad M \neq \texttt{fail}^{\widetilde{y}}}{M[x_1, \cdots, x_k \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \displaystyle\sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket}$$

and $\mathbb{M}' = \sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket$ To simplify the proof we take $k = 2$, as the case $k > 2$ is similar. Therefore,

- $C = \wr N_1 \int \cdot \wr N_2 \int$; and
- $\mathsf{PER}(C) = \{\wr N_1 \int \cdot \wr N_2 \int, \wr N_2 \int \cdot \wr N_1 \int\}$.

Since $\Theta;\Gamma \models \mathbb{M} : \tau$ we get a derivation: (we omit the labels [FS : ex-sub] and [FS:share])

$$\dfrac{\dfrac{\Theta, x^! : \eta; \Gamma', x_1 : \sigma, x_2 : \sigma \models M : \tau \quad x \notin \mathtt{dom}(\Gamma) \quad k \neq 0}{\Theta, x^! : \eta; \Gamma', x : \sigma^2 \models M[\widetilde{x} \leftarrow x] : \tau} \quad \Theta;\Delta \models B : (\sigma^k, \epsilon) \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle : \tau}$$

where $\Gamma = \Gamma', \Delta$. Consider the wf derivation for $\Pi_{1,2}$: (we omit the labels [FS : ex-sub$^!$] and [FS : ex-sub$^\ell$])

$$\dfrac{\dfrac{\dfrac{\Theta, x^! : \eta; \Gamma', x_1 : \sigma, x_2 : \sigma \models M : \tau \quad \Theta;\Delta_1 \models N_1 : \sigma}{\Theta, x^! : \eta; \Gamma', x_2 : \sigma, \Delta_1 \models M\langle\!| N_1/x_1 |\!\rangle : \tau} \quad \Theta;\Delta_2 \models N_2 : \sigma}{\Theta, x^! : \eta; \Gamma', \Delta \models M\langle\!| N_1/x_1 |\!\rangle\langle\!| N_2/x_2 |\!\rangle : \tau} \quad \Theta; \text{-} \models U : \epsilon \quad \eta \propto \epsilon}{\Theta;\Gamma',\Delta \models M\langle\!| N_1/x_1 |\!\rangle\langle\!| N_2/x_2 |\!\rangle\lfloor\!\lfloor U/x \rceil\!\rceil : \tau}$$

Similarly, we can obtain a derivation $\Pi_{2,1}$ of $\Theta;\Gamma',\Delta \models M\langle\!| N_2/x_1 |\!\rangle\langle\!| N_1/x_2 |\!\rangle\lfloor\!\lfloor U/x \rceil\!\rceil : \tau$. Finally, applying [FS:sum]:

$$[\mathtt{FS{:}sum}]\ \dfrac{\Pi_{1,2} \quad \Pi_{2,1}}{\Theta;\Gamma',\Delta \models M\langle\!| N_1/x_1 |\!\rangle\langle\!| N_2/x_2 |\!\rangle\lfloor\!\lfloor U/x \rceil\!\rceil + M\langle\!| N_2/x_1 |\!\rangle\langle\!| N_1/x_2 |\!\rangle\lfloor\!\lfloor U/x \rceil\!\rceil : \tau}$$

and the result follows.

3. Rule [RS:Fetch$^\ell$].
Then $\mathbb{M} = M\langle\!| N/x |\!\rangle$ where $\mathsf{head}(M) = x$. The reduction is:

$$[\mathtt{RS{:}Fetch}^\ell]\ \dfrac{\mathsf{head}(M) = x}{M\langle\!| N/x |\!\rangle \longrightarrow M\{\!| N/x |\!\}}$$

and $\mathbb{M}' = M\langle\!| N/x |\!\rangle$. Since $\Theta;\Gamma \models \mathbb{M} : \tau$ we get the following derivation:

$$[\mathtt{FS{:}ex\text{-}sub}^\ell]\ \dfrac{\Theta;\Gamma', x : \sigma \models M : \tau \quad \Theta;\Delta \models N : \sigma}{\Theta;\Gamma',\Delta \models M\langle\!| N/x |\!\rangle : \tau}$$

where $\Gamma = \Gamma', \Delta$. By Lemma 48, we obtain the derivation $\Theta;\Gamma',\Delta \models M\{\!| N/x |\!\} : \tau$.

4. Rule [RS:Fetch$^!$].
Then $\mathbb{M} = M\lfloor\!\lfloor U/x \rceil\!\rceil$ where $\mathsf{head}(M) = x[i]$. The reduction is:

$$\dfrac{\mathsf{head}(M) = x[i] \quad U_i = \wr N \int^!}{M\lfloor\!\lfloor U/x \rceil\!\rceil \longrightarrow M\{\!| N/x[i] |\!\}\lfloor\!\lfloor U/x \rceil\!\rceil}$$

and $\mathbb{M}' = M\lfloor\!\lfloor U/x \rceil\!\rceil$. Since $\Theta;\Gamma \models \mathbb{M} : \tau$ we get the following derivation:

$$[\mathtt{FS{:}ex\text{-}sub}^!]\ \dfrac{\Theta, x^! : \eta; \Gamma \models M : \tau \quad \Theta; \text{-} \models U : \epsilon \quad \eta \propto \epsilon}{\Theta;\Gamma \models M\lfloor\!\lfloor U/x \rceil\!\rceil : \tau}$$

By Lemma 49, we obtain the derivation $\Theta;\Gamma \models M\{\!| N/x[i] |\!\}\lfloor\!\lfloor U/x \rceil\!\rceil : \tau$.

5. Rule [RS:TCont].
Then $\mathbb{M} = C[M]$ and the reduction is as follows:

$$[\mathtt{RS{:}TCont}]\ \dfrac{M \longrightarrow M_1' + \cdots + M_k'}{C[M] \longrightarrow C[M_1'] + \cdots + C[M_k']}$$

with $\mathbb{M}' = C[M] \longrightarrow C[M_1'] + \cdots + C[M_k']$. The proof proceeds by analysing the context C. There are four cases:

a. $C = [\cdot]\ B$.
In this case $\mathbb{M} = M\ B$, for some B. Since $\Gamma \vdash \mathbb{M} : \tau$ one has a derivation:

$$[\texttt{FS:app}]\ \frac{\Theta; \Gamma' \models M : (\sigma^j, \eta) \to \tau \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M\ B : \tau}$$

where $\Gamma = \Gamma', \Delta$. From $\Gamma' \models M : \sigma^j \to \tau$ and the reduction $M \longrightarrow M'_1 + \cdots + M'_k$, one has by IH that $\Gamma' \models M'_1 + \ldots, M'_k : \sigma^j \to \tau$, which entails $\Gamma' \models M'_i : \sigma^j \to \tau$, for $i = 1, \ldots, k$, via rule [FS:sum]. Finally, we may type the following:

$$[\texttt{FS:sum}]\ \frac{\forall i \in 1, \cdots, l \qquad [\texttt{FS:app}]\ \dfrac{\Theta; \Gamma' \models M'_i : (\sigma^j, \eta) \to \tau \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M'_i\ B : \tau}}{\Gamma', \Delta \models (M'_1\ B) + \cdots + (M'_l\ B) : \tau}$$

Since $\mathbb{M}' = (C[M'_1]) + \cdots + (C[M'_l]) = M'_1 B + \ldots + M'_k B$, the result follows.

b. Cases $C = [\cdot]\langle\!|N/x|\!\rangle$ and $C = [\cdot][\widetilde{x} \leftarrow x]$ are similar to the previous.

c. Other cases proceed similarly.

6. Rule [RS:ECont].
This case is analogous to the previous.

7. Rule [$\texttt{RS:Fail}^{\ell}$].
Then $\mathbb{M} = M[\widetilde{x} \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle$ where $C = \wr N_1 \int \cdots \cdot \wr N_l \int$ and the reduction is:

$$[\texttt{RS:Fail}^{\ell}]\ \frac{k \neq \mathsf{size}(C) \qquad \widetilde{y} = (\mathsf{lfv}(M) \setminus \{\widetilde{x}\}) \cup \mathsf{lfv}(C)}{M[x_1, \cdots, x_k \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \sum_{C_i \in \mathsf{PER}(C)} \texttt{fail}^{\widetilde{y}}}$$

where $\mathbb{M}' = \sum_{C_i \in \mathsf{PER}(C)} \texttt{fail}^{\widetilde{y}}$. Since $\Theta, x : \eta; \Gamma', x_1 : \sigma, \ldots, x_k : \sigma \models \mathbb{M}$, one has a derivation:

$$\frac{[\texttt{FS:ex-sub}]\ \dfrac{\Theta, x : \eta; \Gamma', x_1 : \sigma, \ldots, x_k : \sigma \models M : \tau}{\Theta, x : \eta; \Gamma', x : \sigma^k \models M[x_1, \cdots, x_k \leftarrow x] : \tau} \qquad \Theta; \Delta \models C \star U : (\sigma^j, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models M[x_1, \cdots, x_k] \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle : \tau}\ [\texttt{FS:ex-sub}]$$

where $\Gamma = \Gamma', \Delta$. We may type the following:

$$[\texttt{FS:fail}]\ \frac{}{\Theta; \Gamma', \Delta \models \texttt{fail}^{\widetilde{y}} : \tau}$$

since Γ', Δ contain assignments on the free variables in M and B. Therefore, $\Theta; \Gamma \models \texttt{fail}^{\widetilde{y}} : \tau$, by applying [FS:sum], it follows that $\Theta; \Gamma \models \sum_{B_i \in \mathsf{PER}(B)} \texttt{fail}^{\widetilde{y}} : \tau$,as required.

8. Rule [$\texttt{RS:Fail}^!$].
Then $M \llbracket U/x \rrbracket$ where $\mathsf{head}(M) = x[i]$ and $B = U_i = 1^!$ and the reduction is:

$$[\texttt{RS:Fail}^!]\ \frac{\mathsf{head}(M) = x[i] \qquad U_i = 1^! \qquad \widetilde{y} = \mathsf{lfv}(M)}{M \llbracket U/x \rrbracket \longrightarrow M\{\!|\texttt{fail}^{\emptyset}/x[i]|\!\}\llbracket U/x \rrbracket}$$

with $\mathbb{M}' = M\{\!|\texttt{fail}^{\emptyset}/x[i]|\!\}\llbracket U/x \rrbracket$. By hypothesis, one has the derivation:

$$[\texttt{FS:Esub}^!]\ \frac{\Theta, x : \eta; \Gamma \models M : \tau \qquad \Theta; \text{-} \models U : \epsilon \qquad \eta \propto \epsilon}{\Theta; \Gamma \models M \llbracket U/x \rrbracket : \tau}$$

By Lemma 49, there exists a derivation Π_1 of $\Theta, x^! : \eta; \Gamma' \models M\{\!|\texttt{fail}^{\emptyset}/x[i]|\!\} : \tau$. Thus,

$$[\texttt{FS:Esub}^!]\ \frac{\Theta, x^! : \eta; \Gamma \models M\{\!|\texttt{fail}^{\emptyset}/x[i]|\!\} : \tau \qquad \Theta; \text{-} \models U : \epsilon \qquad \eta \propto \epsilon}{\Theta; \Gamma \models M\{\!|\texttt{fail}^{\emptyset}/x[i]|\!\}\llbracket U/x \rrbracket : \tau}$$

9. Rule $[\texttt{RS:Cons}_1]$.
Then $\mathbb{M} = \texttt{fail}^{\widetilde{x}}\ B$ where $B = \wr N_1 \int \cdots \cdot \wr N_k \int$ and the reduction is:

$$[\texttt{RS:Cons}_1]\ \frac{\widetilde{y} = \mathsf{lfv}(C)}{\texttt{fail}^{\widetilde{x}}\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \texttt{fail}^{\widetilde{x} \uplus \widetilde{y}}}$$

and $\mathbb{M}' = \sum_{\mathsf{PER}(B)} \texttt{fail}^{\widetilde{x} \cup \widetilde{y}}$. Since $\Gamma \models \mathbb{M} : \tau$, one has the derivation:

$$[\texttt{F:app}]\ \frac{[\texttt{F:fail}]\ \frac{}{\Theta; \Gamma' \models \texttt{fail}^{\widetilde{x}} : (\sigma^j, \eta) \to \tau} \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma', \Delta \models \texttt{fail}^{\widetilde{x}}\ B : \tau}$$

Hence $\Gamma = \Gamma', \Delta$ and we may type the following:

$$[\texttt{F:sum}]\ \frac{[\texttt{F:fail}]\ \frac{}{\Theta; \Gamma \models \texttt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau} \quad \cdots \quad [\texttt{F:fail}]\ \frac{}{\Theta; \Gamma \models \texttt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau}}{\Theta; \Gamma \models \sum_{\mathsf{PER}(C)} \texttt{fail}^{\widetilde{x} \uplus \widetilde{y}} : \tau}$$

The proof for the cases of $[\texttt{RS:Cons}_2]$, $[\texttt{RS:Cons}_3]$ and $[\texttt{RS:Cons}_4]$ proceed similarly

◀

E Appendix to Subsection 4.3

E.1 Encodability Criteria

We follow the criteria in [9], a widely studied abstract framework for establishing the *quality* of encodings. A *language* $\mathcal{L}$ is a pair: a set of terms and a reduction semantics $\longrightarrow$ on terms (with reflexive, transitive closure denoted $\xrightarrow{*}$). A correct encoding translates terms of a source language $\mathcal{L}_1 = (\mathcal{M}, \longrightarrow_1)$ into terms of a target language $\mathcal{L}_2(\mathcal{P}, \longrightarrow_2)$ by respecting certain criteria. The criteria in [9] concern *untyped* languages; because we treat *typed* languages, we follow [14] in requiring that encodings satisfy the following criteria:

1. **Type preservation:** For every well-typed M, it holds that $\llbracket M \rrbracket$ is well-typed.
2. **Operational Completeness:** For every M, M' such that $M \xrightarrow{*}_1 M'$, it holds that $\llbracket M \rrbracket \xrightarrow{*}_2 \approx_2 \llbracket M' \rrbracket$.
3. **Operational Soundness:** For every M and P such that $\llbracket M \rrbracket \xrightarrow{*}_2 P$, there exists an M' such that $M \longrightarrow_1^* M'$ and $P \xrightarrow{*}_2 \approx_2 \llbracket M' \rrbracket$.
4. **Success Sensitiveness:** For every M, it holds that $M\checkmark_1$ if and only if $\llbracket M \rrbracket \checkmark_2$, where $\checkmark_1$ and $\checkmark_2$ denote a success predicate in $\mathcal{M}$ and $\mathcal{P}$, respectively.

In addition to these semantic criteria, we shall also consider *compositionality*: a composite source term is encoded as the combination of the encodings of its sub-terms. Success sensitiveness complements completeness and soundness, giving information about observable behaviors. The so-called success predicates $\checkmark_1$ and $\checkmark_2$ serve as a minimal notion of *observables*; the criterion then says that observability of success of a source term implies observability of success in the corresponding target term, and vice-versa.

E.2 Correctness of $(\!| \cdot |\!)^{\circ}$

The correctness of the encoding from $(\!|\cdot|\!)^{\circ}$ from $u\lambda^{\lightning}_{\oplus}$ to $u\widehat{\lambda}^{\lightning}_{\oplus}$ relies on an encoding on contexts (Def. 51), auxiliary propositions (Propositions 52 and 53) for well-formedness preservation (Theorema 54), operational soundness (Theorem 56) and completeness (Theorem 57), and success sensitivity (Theorem 61).

▶ **Definition 51** (Encoding on Contexts). *We define an encoding* $\{\!\{\cdot\}\!\}$ *on contexts:*

$$\{\!\{\Theta\}\!\} = \Theta \qquad \{\!\{\emptyset\}\!\} = \emptyset$$
$$\{\!\{x:\tau,\Gamma\}\!\} = x:\tau,\{\!\{\Gamma\}\!\} \qquad (x \notin \mathtt{dom}(\Gamma))$$
$$\{\!\{x:\tau,\cdots,x:\tau,\Gamma\}\!\} = x:\tau\wedge\cdots\wedge\tau,\{\!\{\Gamma\}\!\} \qquad (x \notin \mathtt{dom}(\Gamma))$$

▶ **Proposition 52.** *Let M, N be terms. We have:*
1. $(\!|M\{\!|N/x|\!\}|\!)^{\bullet} = (\!|M|\!)^{\bullet}\{\!|(\!|N|\!)^{\bullet}/x|\!\}$.
2. $(\!|M\langle \widetilde{x}/x\rangle|\!)^{\bullet} = (\!|M|\!)^{\bullet}\langle \widetilde{x}/x\rangle$, *where $\widetilde{x} = x_1,\ldots,x_k$ is sequence of pairwise distinct fresh variables.*

Proof. By induction of the structure of M. ◀

▶ **Proposition 53** (Well-formedness Preservation under Linear Substitutions in $u\lambda^{\lightning}_{\oplus}$). *Let $M \in \lambda^{\lightning}_{\oplus}$. If $\Theta;\Gamma,x:\sigma \models M:\tau$ and $\Theta;\Gamma \models x_i:\sigma$ then $\Theta;\Gamma,x_i:\sigma \models M\langle x_i/x\rangle:\tau$.*

Proof. Standard, by induction on the well-formedness derivation rules in Fig. 3. ◀

▶ **Proposition 54** (Well-formedness preservation for $(\!| - |\!)^{\bullet}$). *Let B and $\mathbb{M}$ be a bag and a expression in $u\lambda^{\lightning}_{\oplus}$, respectively.*
1. *If $\Theta;\Gamma \models B : (\sigma^k,\eta)$ and $\mathtt{dom}(\Gamma) = \mathsf{mlfv}(B)$ then $\{\!\{\Theta\}\!\};\{\!\{\Gamma\}\!\} \models (\!|B|\!)^{\bullet} : (\sigma^k,\eta)$ and $\forall\, x:\pi\in\Gamma,\ \pi=\tau$ for some τ.*
2. *If $\Theta;\Gamma \models \mathbb{M}:\sigma$ and $\mathtt{dom}(\Gamma) = \mathsf{mlfv}(\mathbb{M})$ then $\{\!\{\Theta\}\!\};\{\!\{\Gamma\}\!\} \models (\!|\mathbb{M}|\!)^{\bullet}:\sigma$ and $\forall\, x:\pi\in\Gamma,\ \pi=\tau$ for some τ.*

▶ **Theorem 55** (Well-formedness Preservation for $(\!| - |\!)^{\circ}$). *Let B and $\mathbb{M}$ be a bag and an expression in $\lambda^{\lightning}_{\oplus}$, respectively.*
1. *If $\Theta;\Gamma \models B:(\sigma^k,\eta)$ and $\mathtt{dom}(\Gamma) = \mathsf{fv}(B)$ then $\{\!\{\Theta\}\!\};\{\!\{\Gamma\}\!\} \models (\!|B|\!)^{\circ}:(\sigma^k,\eta)$.*
2. *If $\Theta;\Gamma \models \mathbb{M}:\sigma$ and $\mathtt{dom}(\Gamma) = \mathsf{fv}(\mathbb{M})$ then $\{\!\{\Theta\}\!\};\{\!\{\Gamma\}\!\} \models (\!|\mathbb{M}|\!)^{\circ}:\sigma$.*

Proof. By mutual induction on the typing derivations $\Theta;\Gamma \models B:(\sigma^k,\eta)$ and $\Theta;\Gamma \models \mathbb{M}:\sigma$, exploiting Proposition 54. The analysis for bags Part 1. follows directly from the IHs and will be omitted. As for Part 2. there are two main cases to consider:
1. $\mathbb{M} = M$.
   Without loss of generality, assume $\mathsf{fv}(M) = \{x,y\}$. Then, $\Theta;\hat{x}:\sigma_1^j,\hat{y}:\sigma_2^k \models M:\tau$ where $\#(x,M) = j$ and $\#(y,M) = k$, for some positive integers j and k.
   After $j+k$ applications of Proposition 53 we obtain:

$$\Theta; x_1:\sigma_1,\cdots,x_j:\sigma_1,y_1:\sigma_2,\cdots,y_k:\sigma_2 \models M\langle\widetilde{x}/x\rangle\langle\widetilde{y}/y\rangle:\tau$$

   where $\widetilde{x} = x_1,\cdots,x_j$ and $\widetilde{y} = y_1,\cdots,y_k$. From Proposition 54 one has

$$\{\!\{\Theta\}\!\};\{\!\{x_1:\sigma_1,\cdots,x_j:\sigma_1,y_1:\sigma_2,\cdots,y_k:\sigma_2\}\!\} \models (\!|M\langle\widetilde{x}/x\rangle\langle\widetilde{y}/y\rangle|\!)^{\bullet}:\tau$$

   Since $\{\!\{x_1:\sigma_1,\cdots,x_j:\sigma_1,y_1:\sigma_2,\cdots,y_k:\sigma_2\}\!\} = x_1:\sigma_1,\cdots,x_j:\sigma_1,y_1:\sigma_2,\cdots,y_k:\sigma_2$ and $\{\!\{\Theta\}\!\} = \Theta$, we have the following derivation:

$$
[\texttt{FS : share}]\ \cfrac{[\texttt{FS : share}]\ \cfrac{\Theta; x_1 : \sigma_1, \cdots, x_j : \sigma_1, y_1 : \sigma_2, \cdots, y_k : \sigma_2 \models \llparenthesis M\langle \widetilde{x}/x\rangle\langle \widetilde{y}/y\rangle \rrparenthesis^{\bullet} : \tau}{\Theta; x : \sigma_1^j, y_1 : \sigma_2, \cdots, y_k : \sigma_2 \models \llparenthesis M\langle \widetilde{x}/x\rangle\langle \widetilde{y}/y\rangle \rrparenthesis^{\bullet}[\widetilde{x} \leftarrow x] : \tau}}{\Theta; x : \sigma_1^j, y : \sigma_2^k \models \llparenthesis M\langle \widetilde{x}/x\rangle\langle \widetilde{y}/y\rangle \rrparenthesis^{\bullet}[\widetilde{x} \leftarrow x][\widetilde{y} \leftarrow y] : \tau}
$$

By expanding Def. 24, we have

$$
\llparenthesis M \rrparenthesis^{\circ} = \llparenthesis M\langle \widetilde{x}/x\rangle\langle \widetilde{y}/y\rangle \rrparenthesis^{\bullet}[\widetilde{x} \leftarrow x][\widetilde{y} \leftarrow y]
$$

which completes the proof for this case.

2. $\mathbb{M} = M_1 + \cdots + M_n$:
   This case proceeds easily by IH, using Rule [$\texttt{FS : sum}$].

◀

▶ **Theorem 56** (Operational Completeness). *Let $\mathbb{M}, \mathbb{N}$ be well-formed $\lambda^{\lightning}_{\oplus}$ expressions. Suppose $\mathbb{N} \longrightarrow_{[\mathtt{R}]} \mathbb{M}$.*

1. *If $[\mathtt{R}] = [\mathtt{R : Beta}]$ then $\llparenthesis \mathbb{N} \rrparenthesis^{\circ} \longrightarrow^{\leq 2} \llparenthesis \mathbb{M} \rrparenthesis^{\circ}$;*
2. *If $[\mathtt{R}] = [\mathtt{R : Fetch}]$ then $\llparenthesis \mathbb{N} \rrparenthesis^{\circ} \longrightarrow^{+} \llparenthesis \mathbb{M}' \rrparenthesis^{\circ}$, for some $\mathbb{M}$.*
3. *If $[\mathtt{R}] \neq [\mathtt{R : Beta}]$ and $[\mathtt{R}] \neq [\mathtt{R : Fetch}]$ then $\llparenthesis \mathbb{N} \rrparenthesis^{\circ} \longrightarrow \llparenthesis \mathbb{M} \rrparenthesis^{\circ}$.*

Proof. We proceed by induction on the the rule from Fig. 1 applied to infer $\mathbb{N} \longrightarrow \mathbb{M}$, distinguishing the three cases: (below $[\widetilde{x_1 \leftarrow x_n}]$ abbreviates $[\widetilde{x_1} \leftarrow x_1] \cdots [\widetilde{x_n} \leftarrow x_n]$).

1. The rule applied is $[\mathtt{R}] = [\mathtt{R : Beta}]$.
   In this case, $\mathbb{N} = (\lambda x.M')B$, where $B = C \star U$, the reduction is

$$
[\mathtt{R : Beta}]\ \cfrac{}{(\lambda x.M)B \longrightarrow M\ \langle\!\langle B/x \rangle\!\rangle}
$$

   and $\mathbb{M} = M'\langle\!\langle B/x \rangle\!\rangle$. Below we assume $\mathsf{lfv}(\mathbb{N}) = \{x_1, \ldots, x_k\}$ and $\widetilde{x_i} = x_{i_1}, \ldots, x_{i_{j_i}}$, where $j_i = \#(x_i, N)$, for $1 \leq i \leq k$. On the one hand, we have:

$$
\begin{aligned}
\llparenthesis \mathbb{N} \rrparenthesis^{\circ} &= \llparenthesis (\lambda x.M')B \rrparenthesis^{\circ} = \llparenthesis ((\lambda x.M')B)\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_k}/x_k\rangle \rrparenthesis^{\bullet}[\widetilde{x_1 \leftarrow x_k}] \\
&= \llparenthesis (\lambda x.M'')B' \rrparenthesis^{\bullet}[\widetilde{x_1 \leftarrow x_k}] = (\llparenthesis \lambda x.M'' \rrparenthesis^{\bullet} \llparenthesis B' \rrparenthesis^{\bullet})[\widetilde{x_1 \leftarrow x_k}] \\
&= ((\lambda x.\llparenthesis M''\langle \widetilde{y}/x\rangle \rrparenthesis^{\bullet}[\widetilde{y} \leftarrow x]) \llparenthesis B' \rrparenthesis^{\bullet})[\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow_{[\mathtt{RS:Beta}]} (\llparenthesis M''\langle \widetilde{y}/x\rangle \rrparenthesis^{\bullet}[\widetilde{y} \leftarrow x]\langle\!\langle \llparenthesis B' \rrparenthesis^{\bullet}/x \rangle\!\rangle)[\widetilde{x_1 \leftarrow x_k}] = \mathbb{L}
\end{aligned}
\tag{1}
$$

   On the other hand, we have:

$$
\begin{aligned}
\llparenthesis \mathbb{M} \rrparenthesis^{\circ} &= \llparenthesis M'\langle\!\langle B/x \rangle\!\rangle \rrparenthesis^{\circ} = \llparenthesis M'\langle\!\langle B/x \rangle\!\rangle\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_k}/x_k\rangle \rrparenthesis^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= \llparenthesis M''\langle\!\langle B'/x \rangle\!\rangle \rrparenthesis^{\bullet}[\widetilde{x_1 \leftarrow x_k}]
\end{aligned}
\tag{2}
$$

   We need to analyze two sub-cases: either $\#(x, M) = \mathsf{size}(C)$ or $\#(x, M) = k \geq 0$ and our first sub-case is not met.

   a. If $\#(x, M) = \mathsf{size}(C)$ then we can reduce $\mathbb{L}$ as: (via [$\mathtt{RS : Ex - sub}$])

$$
\mathbb{L} \longrightarrow \sum_{C_i \in \mathsf{PER}(\llparenthesis C \rrparenthesis^{\bullet})} \llparenthesis M''\langle \widetilde{y}/x\rangle \rrparenthesis^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(n)/y_n |\!\rangle [\![U/x]\!] [\widetilde{x_1 \leftarrow x_k}] = \llparenthesis \mathbb{M} \rrparenthesis^{\circ}
$$

   From (1) and (2) and $\widetilde{y} = y_1 \ldots y_n$, one has the result.

   b. Otherwise, $\#(x, M) = n \geq 0$.
   Expanding the encoding in (2) :

$$
\llparenthesis M \rrparenthesis^{\circ} = \llparenthesis M''\langle\!\langle B'/x \rangle\!\rangle \rrparenthesis^{\bullet}[\widetilde{x_1 \leftarrow x_k}] = (\llparenthesis M''\langle \widetilde{y}/x\rangle \rrparenthesis^{\bullet}[\widetilde{y} \leftarrow x]\langle\!\langle \llparenthesis B' \rrparenthesis^{\bullet}/x \rangle\!\rangle)[\widetilde{x_1 \leftarrow x_k}]
$$

   Therefore $\llparenthesis M \rrparenthesis^{\circ} = \mathbb{L}$ and $\llparenthesis \mathbb{N} \rrparenthesis^{\circ} \longrightarrow \llparenthesis \mathbb{M} \rrparenthesis^{\circ}$.

2. The rule applied is $[\mathtt{R}] = [\mathtt{R} : \mathtt{Fetch}^{\ell}]$.
Then $\mathbb{N} = M\langle\!\langle C \star U / x \rangle\!\rangle$ and the reduction is

$$[\mathtt{R} : \mathtt{Fetch}^{\ell}]\ \frac{\mathsf{head}(M) = x \quad C = \wr N_1 \int \cdot \cdots \cdot \wr N_k \int\ ,\ k \geq 1 \quad \#(x, M) = k}{M\langle\!\langle C \star U / x \rangle\!\rangle \longrightarrow M\{\!| N_1/x |\!\}\langle\!\langle (C \setminus N_1) \star U / x \rangle\!\rangle + \cdots + M\{\!| N_k/x |\!\}\langle\!\langle (C \setminus N_k) \star U / x \rangle\!\rangle}$$

with $\mathbb{M} = M\{\!| N_1/x |\!\}\langle\!\langle (C \setminus N_1) \star U / x \rangle\!\rangle + \cdots + M\{\!| N_k/x |\!\}\langle\!\langle (C \setminus N_k) \star U / x \rangle\!\rangle$.
Below we assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_k\}$ and $\widetilde{x_i} = x_{i_1}, \ldots, x_{i_{j_i}}$, where $j_i = \#(x_i, N)$, for $1 \leq i \leq k$. On the one hand, we have: (last rule is $[\mathtt{RS{:}Fetch}^{\ell}]$)

$$\begin{aligned}
(\!|\mathbb{N}|\!)^{\circ} &= (\!| M\langle\!\langle C \star U / x \rangle\!\rangle |\!)^{\circ} = (\!| M\langle\!\langle C \star U / x \rangle\!\rangle \langle \widetilde{x_1}/x_1 \rangle \cdots \langle \widetilde{x_k}/x_k \rangle |\!)^{\bullet}[\widetilde{x_1 \leftarrow x_k}] \\
&= (\!| M'\langle\!\langle C' \star U / x \rangle\!\rangle |\!)^{\bullet}[\widetilde{x_1 \leftarrow x_k}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M'\langle \widetilde{y}/x \rangle |\!)^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M'' |\!)^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M''\{\!| C_i(1)/y_1 |\!\} |\!)^{\bullet} \langle\!| C_i(2)/y_2 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&= \mathbb{L}
\end{aligned}$$

We assume for simplicity that $\mathsf{head}(M'') = y_1$ On the other hand, we have:

$$\begin{aligned}
(\!|\mathbb{M}|\!)^{\circ} &= (\!| M\{\!| N_1/x |\!\}\langle\!\langle (C \setminus N_1) \star U / x \rangle\!\rangle + \cdots + M\{\!| N_k/x |\!\}\langle\!\langle (C \setminus N_k) \star U / x \rangle\!\rangle |\!)^{\circ} \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M''\{\!| C_i(1)/y_1 |\!\} |\!)^{\bullet} \langle\!| C_i(2)/y_2 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&= \mathbb{L}
\end{aligned}$$

From these developments from $(\!|\mathbb{N}|\!)^{\circ}$ and $(\!|\mathbb{M}|\!)^{\circ}$, and $\widetilde{y} = y_1 \ldots y_n$, one has the result.

3. The rule applied is $[\mathtt{R}] = [\mathtt{R} : \mathtt{Fetch}^{!}]$.
Then $\mathbb{N} = M\langle\!\langle C \star U / x \rangle\!\rangle$ and the reduction is

$$[\mathtt{R} : \mathtt{Fetch}^{!}]\ \frac{\mathsf{head}(M) = x[i] \quad U_i = \wr N \int^{!}}{M\ \langle\!\langle C \star U / x \rangle\!\rangle \longrightarrow M\{\!| N/x[i] |\!\}\langle\!\langle C \star U / x \rangle\!\rangle}$$

with $\mathbb{M} = M\{\!| N/x[i] |\!\}\langle\!\langle C \star U / x \rangle\!\rangle$. Below we assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_k\}$ and $\widetilde{x_i} = x_{i_1}, \ldots, x_{i_{j_i}}$, where $j_i = \#(x_i, N)$, for $1 \leq i \leq k$.
On the one hand, we have: (the last rule is $[\mathtt{RS} : \mathtt{Fetch}^{!}]$)

$$\begin{aligned}
(\!|\mathbb{N}|\!)^{\circ} &= (\!| M\langle\!\langle C \star U / x \rangle\!\rangle |\!)^{\circ} = (\!| M\langle\!\langle C \star U / x \rangle\!\rangle \langle \widetilde{x_1}/x_1 \rangle \cdots \langle \widetilde{x_k}/x_k \rangle |\!)^{\bullet}[\widetilde{x_1 \leftarrow x_k}] \\
&= (\!| M'\langle\!\langle C' \star U / x \rangle\!\rangle |\!)^{\bullet}[\widetilde{x_1 \leftarrow x_k}]] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M'\langle \widetilde{y}/x \rangle |\!)^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M'' |\!)^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M''\{\!| N/x[i] |\!\} |\!)^{\bullet} \langle\!| C_i(2)/y_2 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] = \mathbb{L}
\end{aligned} \tag{3}$$

On the other hand, assuming for simplicity that $\mathsf{head}(M'') = x[i]$ and $U_i = N$, we have

$$\begin{aligned}(\!|\mathbb{M}|\!)^\circ &= (\!|M\{\!|N/x[i]|\!\}\langle\!\langle C \star U/x\rangle\!\rangle|\!)^\circ \\ &= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} ((\!|M''\{\!|N/x[i]|\!\}|\!)^\bullet \langle\!|C_i(2)/y_2|\!\rangle \cdots \langle\!|C_i(k)/y_k|\!\rangle \llbracket U/x \rrbracket)[\widetilde{x_1 \leftarrow x_k}] = \mathbb{L} \qquad (4)\end{aligned}$$

From (3) and (4), one has the result.

4. The rule applied is $[\mathtt{R}] \neq [\mathtt{R : Beta}]$ and $[\mathtt{R}] \neq [\mathtt{R : Fetch}]$. There are two possible cases:

a. $[\mathtt{R}] = [\mathtt{R : Fail}^\ell]$
Then $\mathbb{N} = M\langle\!\langle C \star U/x\rangle\!\rangle$ and the reduction is

$$[\mathtt{R : Fail}^\ell]\ \frac{\#(x, M) \neq \mathsf{size}(C) \qquad \widetilde{z} = (\mathsf{mlfv}(M) \setminus x) \uplus \mathsf{mlfv}(C)}{M\langle\!\langle C \star U/x\rangle\!\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}}}$$

where $\mathbb{M} = \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}}$. Below assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_n\}$.
On the one hand, we have:

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ &= (\!|M\langle\!\langle C \star U/x\rangle\!\rangle|\!)^\circ = (\!|M\langle\!\langle C \star U/x\rangle\!\rangle\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_n}/x_n\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] \\ &= (\!|M'\langle\!\langle C' \star U/x\rangle\!\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] \\ &= (\!|M'\langle y_1, \cdots, y_k/x\rangle|\!)^\bullet[y_1, \cdots, y_k \leftarrow x]\langle\!\langle C' \star U/x\rangle\!\rangle[\widetilde{x_1 \leftarrow x_n}] \\ &\longrightarrow_{[\mathtt{RS:Fail}^\ell]} \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}, \widetilde{x_1}, \cdots, \widetilde{x_n}}[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}\end{aligned}$$

On the other hand, we have:

$$(\!|\mathbb{M}|\!)^\circ = \sum_{\mathsf{PER}(C)} (\!|\mathtt{fail}^{\widetilde{z}}|\!)^\circ = \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}, \widetilde{x_1}, \cdots, \widetilde{x_n}}[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}$$

Therefore, $(\!|\mathbb{N}|\!)^\circ \longrightarrow (\!|\mathbb{M}|\!)^\circ$ and the result follows.

b. $[\mathtt{R}] = [\mathtt{R : Fail}^!]$
Then $\mathbb{N} = M\langle\!\langle C \star U/x\rangle\!\rangle$ and the reduction is

$$[\mathtt{R : Fail}^!]\ \frac{\#(x, M) = \mathsf{size}(C) \qquad U_i = 1^! \qquad \mathsf{head}(M) = x[i]}{M\langle\!\langle C \star U/x\rangle\!\rangle \longrightarrow M\{\!|\mathtt{fail}^\emptyset/x[i]|\!\}\langle\!\langle C \star U/x\rangle\!\rangle}$$

where $\mathbb{M} = M\{\!|\mathtt{fail}^\emptyset/x[i]|\!\}\langle\!\langle C \star U/x\rangle\!\rangle$.
Below we assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_k\}$ and $\widetilde{x_i} = x_{i_1}, \ldots, x_{i_{j_i}}$, where $j_i = \#(x_i, N)$, for $1 \leq i \leq k$.
On the one hand, we have: (the last rule applied was $[\mathtt{RS : Fail}^!]$)

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ &= (\!|M\langle\!\langle C \star U/x\rangle\!\rangle|\!)^\circ = (\!|M\langle\!\langle C \star U/x\rangle\!\rangle\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_k}/x_k\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] \\ &= (\!|M'\langle\!\langle C' \star U/x\rangle\!\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] \\ &= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} ((\!|M'\langle \widetilde{y}/x\rangle|\!)^\bullet \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(k)/y_k|\!\rangle \llbracket U/x \rrbracket)[\widetilde{x_1 \leftarrow x_k}] \\ &= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} ((\!|M''|\!)^\bullet \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(k)/y_k|\!\rangle \llbracket U/x \rrbracket)[\widetilde{x_1 \leftarrow x_k}] \\ &\longrightarrow \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} ((\!|M''\{\!|\mathtt{fail}^\emptyset/x[i]|\!\}|\!)^\bullet \langle\!|C_i(2)/y_2|\!\rangle \cdots \langle\!|C_i(k)/y_k|\!\rangle \llbracket U/x \rrbracket)[\widetilde{x_1 \leftarrow x_k}] \\ &= \mathbb{L}\end{aligned}$$

We assume for simplicity that $\mathsf{head}(M'') = x[i]$. On the other hand, we have:

$$\begin{aligned}
(\!|\mathbb{M}|\!)^{\circ} &= (\!| M\{\!|\mathtt{fail}^{\emptyset}/x[i]|\!\}\langle\!\langle C \star U/x\rangle\!\rangle |\!)^{\circ} \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} ((\!| M''\{\!|\mathtt{fail}^{\emptyset}/x[i]|\!\}|\!)^{\bullet} \langle\!| C_i(2)/y_2 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!])[\widetilde{x_1 \leftarrow x_k}] \\
&= \mathbb{L}
\end{aligned}$$

From the $(\!|M|\!)^{\circ}$ and $(\!|N|\!)^{\circ}$ above one has the result.

c. $[\mathtt{R}] = [\mathtt{R}:\mathtt{Cons}_1]$.
Then $\mathbb{N} = (\mathtt{fail}^{\widetilde{z}})\ C \star U$ and the reduction is

$$[\mathtt{R}:\mathtt{Cons}_1]\ \frac{\widetilde{z} = \mathsf{mlfv}(C)}{(\mathtt{fail}^{\widetilde{y}})\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y} \uplus \widetilde{z}}}$$

and $\mathbb{M}' = \sum_{\mathsf{PER}(B)} \mathtt{fail}^{\widetilde{y} \uplus \widetilde{z}}$. Below we assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_n\}$.
On the one hand, we have:

$$\begin{aligned}
(\!|N|\!)^{\circ} &= (\!|\mathtt{fail}^{\widetilde{y}}\ B|\!)^{\circ} = (\!|\mathtt{fail}^{\widetilde{y}}\ C \star U\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_n}/x_n\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= (\!|\mathtt{fail}^{\widetilde{y'}}\ \ C' \star U|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] = (\!|\mathtt{fail}^{\widetilde{y'}}|\!)^{\bullet}\ (\!|C' \star U|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= \mathtt{fail}^{\widetilde{y'}}\ (\!|C' \star U|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \longrightarrow_{[\mathtt{RS}:\mathtt{Cons}_1]} \sum_{\mathsf{PER}(B)} \mathtt{fail}^{\widetilde{y'} \cup \widetilde{z'}}[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}
\end{aligned}$$

Where $\widetilde{y'} \cup \widetilde{z'} = \widetilde{x_1}, \cdots, \widetilde{x_n}$. On the other hand, we have:

$$(\!|M|\!)^{\circ} = \sum_{\mathsf{PER}(B)} (\!|\mathtt{fail}^{\widetilde{y'} \uplus \widetilde{z'}}|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] = \sum_{\mathsf{PER}(B)} \mathtt{fail}^{\widetilde{y'} \cup \widetilde{z'}}[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}$$

Therefore, $(\!|\mathbb{N}|\!)^{\circ} \longrightarrow \mathbb{L} = (\!|\mathbb{M}|\!)^{\circ}$, and the result follows.

d. $[\mathtt{R}] = [\mathtt{R}:\mathtt{Cons}_2]$
Then $\mathbb{N} = \mathtt{fail}^{\widetilde{y}}\ \langle\!\langle C \star U/z\rangle\!\rangle$ and the reduction is

$$[\mathtt{R}:\mathtt{Cons}_2]\ \frac{\#(z, \widetilde{y}) = \mathsf{size}(C) \quad \widetilde{z} = \mathsf{mlfv}(C)}{\mathtt{fail}^{\widetilde{y}}\ \langle\!\langle C \star U/z\rangle\!\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{(\widetilde{x} \setminus z) \uplus \widetilde{z}}}$$

and $\mathbb{M} = \sum_{\mathsf{PER}(C)} \mathtt{fail}^{(\widetilde{y} \setminus x) \uplus \widetilde{z}}$. Below we assume $\mathsf{fv}(\mathbb{N}) = \{x_1, \ldots, x_n\}$.
On the one hand, we have:

$$\begin{aligned}
(\!|\mathbb{N}|\!)^{\circ} &= (\!|\mathtt{fail}^{\widetilde{y}}\ \langle\!\langle C \star U/z\rangle\!\rangle|\!)^{\circ} = (\!|\mathtt{fail}^{\widetilde{y}}\ \langle\!\langle C \star U/z\rangle\!\rangle \langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_n}/x_n\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} (\!|\mathtt{fail}^{\widetilde{y'}}\langle \widetilde{y}/x\rangle|\!)^{\bullet} \langle\!| C_i(1)/y_1 |\!\rangle \cdots \langle\!| C_i(k)/y_k |\!\rangle [\![U/x]\!][\widetilde{x_1 \leftarrow x_n}] \\
&\longrightarrow^{*}_{[\mathtt{RS}:\mathtt{Cons}_3]} \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} (\!|\mathtt{fail}^{(\widetilde{y'} \setminus \widetilde{y}) \uplus \widetilde{z}}|\!)^{\bullet} [\![U/x]\!][\widetilde{x_1 \leftarrow x_n}] \\
&\longrightarrow^{*}_{[\mathtt{RS}:\mathtt{Cons}_4]} \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} (\!|\mathtt{fail}^{(\widetilde{y'} \setminus \widetilde{y}) \uplus \widetilde{z}}|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}]
\end{aligned} \tag{5}$$

As $\widetilde{y}$ consists of free variables, we have that in $\mathtt{fail}^{\widetilde{y}}\ \langle\!\langle C \star U/x\rangle\!\rangle\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_n}/x_n\rangle$ the substitutions also occur on $\widetilde{y}$ resulting in a new $\widetilde{y'}$ where all x_i's are replaced with their fresh components in $\widetilde{x_i}$. Similarly $\widetilde{y''}$ is $\widetilde{y'}$ with each x replaced with a fresh y_i. On the other hand, we have:

$$(\!|M|\!)^\circ = (\!|\sum_{\mathsf{PER}(C)} \mathtt{fail}^{(\widetilde{y}\setminus x)\uplus\widetilde{z}}|\!)^\circ = \sum_{C_i\in\mathsf{PER}((\!|C'|\!)^\bullet)} (\!|\mathtt{fail}^{(\widetilde{y'}\setminus\widetilde{y})\uplus\widetilde{z}}|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] \tag{6}$$

The reductions in (5) and (6) lead to identical expressions.

As before, the reduction via rule [R] could occur inside a context (cf. Rules [$\mathtt{R}:\mathtt{TCont}$] and [$\mathtt{R}:\mathtt{ECont}$]). We consider only the case when the contextual rule used is [$\mathtt{R}:\mathtt{TCont}$]. We have $\mathbb{N} = C[N]$. When we have $C[N] \longrightarrow_{[\mathtt{R}]} C[M]$ such that $N \longrightarrow_{[\mathtt{R}]} M$ we need to show that $(\!|C[N]|\!)^\circ \longrightarrow^j (\!|C[M]|\!)^\circ$for some j dependent on [R]. Firstly let us assume $[\mathtt{R}] = [\mathtt{R}:\mathtt{Cons}_2]$ then we take $j = 1$. Let us take $C[\cdot]$ to be $[\cdot]B$ and $\mathsf{fv}(NB) = \{x_1, \cdots, x_k\}$ then

$$(\!|NB|\!)^\circ = (\!|NB\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_k}/x_k\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] = (\!|N'B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] = (\!|N'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}]$$

We take $N'B' = NB\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_k}/x_k\rangle$, we have by the IH that $(\!|N|\!)^\bullet \longrightarrow (\!|M|\!)^\bullet$ and hence we can deduce that $(\!|N'|\!)^\bullet \longrightarrow (\!|M'|\!)^\bullet$ where $M'B' = MB\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_k}/x_k\rangle$. Finally we have $(\!|N'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] \longrightarrow (\!|M'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}]$ and hence $(\!|C[N]|\!)^\circ \longrightarrow (\!|C[M]|\!)^\circ$. ◀

▶ **Theorem 57** (Operational Soundness). *Let $\mathbb{N}$ be a well-formed $u\lambda^{\lightning}_{\oplus}$ expression. Suppose $(\!|\mathbb{N}|\!)^\circ \longrightarrow \mathbb{L}$. Then, there exists $\mathbb{N}'$ such that $\mathbb{N} \longrightarrow_{[\mathtt{R}]} \mathbb{N}'$ and*

1. *If $[\mathtt{R}] = [\mathtt{R}:\mathtt{Beta}]$ then $\mathbb{L} \longrightarrow^{\leq 1} (\!|\mathbb{N}'|\!)^\circ$;*
2. *If $[\mathtt{R}] \neq [\mathtt{R}:\mathtt{Beta}]$ then $\mathbb{L} \longrightarrow^{*} (\!|\mathbb{N}''|\!)^\circ$, for $\mathbb{N}''$ such that $\mathbb{N}' \equiv_\lambda \mathbb{N}''$.*

Proof. By induction on the structure of $\mathbb{N}$:

1. Cases $\mathbb{N} = x$, $\mathbb{N} = x[i]$, $\mathtt{fail}^{\widetilde{y}}$ and $\mathbb{N} = \lambda x.N$, are trivial, since no reductions can be performed.
2. $\mathbb{N} = NB$:
   Suppose $\mathsf{lfv}(NB) = \{x_1, \cdots, x_n\}$. Then,

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ = (\!|NB|\!)^\circ = (\!|NB\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_n}/x_n\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] &= (\!|N'B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] \\ &= (\!|N'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}]\end{aligned} \tag{7}$$

   where $\widetilde{x_i} = x_{i1}, \ldots, x_{ij_i}$, for $1 \leq i \leq n$. By the reduction rules in Fig. 13 there are three possible reductions starting in $\mathbb{N}$:
   a. $(\!|N'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}]$ reduces via a [$\mathtt{RS}:\mathtt{Beta}$].
   In this case $N = \lambda x.N_1$, and the encoding in (7) gives $N' = N\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_n}/x_n\rangle$, which implies $N' = \lambda x.N_1^{'}$ and the following holds:

$$(\!|N'|\!)^\bullet = (\!|(\lambda x.N_1')|\!)^\bullet = (\lambda x.(\!|N_1'\langle\widetilde{y}/x\rangle|\!)^\bullet[\widetilde{y} \leftarrow x]) = (\lambda x.(\!|N^{''}|\!)^\bullet[\widetilde{y} \leftarrow x])$$

   Thus, we have the following [$\mathtt{RS}:\mathtt{Beta}$] reduction from (7):

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ &= (\!|N'|\!)^\bullet(\!|B'|\!)^\bullet[\widetilde{x_1 \leftarrow x_n}] = (\lambda x.(\!|N''|\!)^\bullet[\widetilde{y} \leftarrow x](\!|B'|\!)^\bullet)[\widetilde{x_1 \leftarrow x_n}] \\ &\longrightarrow_{[\mathtt{RS}:\mathtt{Beta}]} (\!|N^{''}|\!)^\bullet[\widetilde{y} \leftarrow x]\langle\!\langle(\!|B'|\!)^\bullet/x\rangle\!\rangle[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}\end{aligned} \tag{8}$$

Notice that the expression $\mathbb{N}$ can perform the following [$\mathtt{R:Beta}$] reduction:

$$\mathbb{N} = (\lambda x.N_1)B \longrightarrow_{[\mathtt{R:Beta}]} N_1\langle\!\langle B/x\rangle\!\rangle$$

Assuming $\mathbb{N}' = N_1\langle\!\langle B/x\rangle\!\rangle$ and we take $B = C \star U$, there are two cases:

i. $\#(x, M) = \mathsf{size}(C) = k$.
On the one hand,

$$\begin{aligned}
(\!|\mathbb{N}'|\!)^{\circ} &= (\!|N_1\langle\!\langle B/x\rangle\!\rangle|\!)^{\circ} = (\!|N_1\langle\!\langle B/x\rangle\!\rangle\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_n}/x_n\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= (\!|N_1'\langle\!\langle B'/x\rangle\!\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} (\!|N_1'\langle y_1,\cdots,y_k/x\rangle|\!)^{\bullet}\langle\!|C_i(1)/y_1|\!\rangle\cdots\langle\!|C_i(k)/y_k|\!\rangle[\![U/x]\!][\widetilde{x_1 \leftarrow x_n}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^{\bullet})} (\!|N_1''|\!)^{\bullet}\langle\!|C_i(1)/y_1|\!\rangle\cdots\langle\!|C_i(k)/y_k|\!\rangle[\![U/x]\!][\widetilde{x_1 \leftarrow x_n}]
\end{aligned}$$

On the other hand, via application of rule [$\mathtt{RS:Ex\text{-}Sub}$]

$$\begin{aligned}
\mathbb{L} &= (\!|N''|\!)^{\bullet}[\widetilde{y} \leftarrow x]\langle\!\langle (\!|B'|\!)^{\bullet}/x\rangle\!\rangle[\widetilde{x_1 \leftarrow x_n}] \\
&\longrightarrow \sum_{C_i \in \mathsf{PER}((\!|C|\!)^{\bullet})} (\!|N_1''|\!)^{\bullet}\langle\!|C_i(1)/y_1|\!\rangle\cdots\langle\!|C_i(k)/y_k|\!\rangle[\![U/x]\!][\widetilde{x_1 \leftarrow x_n}] = (\!|\mathbb{N}'|\!)^{\circ}
\end{aligned}$$

and the result follows.

ii. Otherwise $\#(x, N_1) \neq \mathsf{size}(C)$.
In this case,

$$\begin{aligned}
(\!|\mathbb{N}'|\!)^{\circ} &= (\!|N_1\langle\!\langle B/x\rangle\!\rangle|\!)^{\circ} = (\!|N_1\langle\!\langle B/x\rangle\!\rangle\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_n}/x_n\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= (\!|N_1'\langle\!\langle B'/x\rangle\!\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] = (\!|N''|\!)^{\bullet}[\widetilde{y} \leftarrow x]\langle\!\langle (\!|B'|\!)^{\bullet}/x\rangle\!\rangle[\widetilde{x_1 \leftarrow x_n}] = \mathbb{L}
\end{aligned}$$

From (8): $(\!|\mathbb{N}|\!)^{\circ} \longrightarrow \mathbb{L} = (\!|\mathbb{N}'|\!)^{\circ}$ and the result follows.

b. $(\!|N'|\!)^{\bullet}(\!|B'|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}]$ reduces via a [$\mathtt{RS:Cons_1}$].
In this case, $N = \mathtt{fail}^{\widetilde{y}}$, and the encoding in (7) gives $N' = N\langle\widetilde{x_1}/x_1\rangle\ldots\langle\widetilde{x_n}/x_n\rangle$, which implies $N' = \mathtt{fail}^{\widetilde{y'}}$, we let $B = C \star U$ and the following:

$$\begin{aligned}
(\!|\mathbb{N}|\!)^{\circ} &= (\!|N'|\!)^{\bullet}(\!|B'|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] = (\!|\mathtt{fail}^{\widetilde{y'}}|\!)^{\bullet}(\!|B'|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \\
&= \mathtt{fail}^{\widetilde{y'}}(\!|B'|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_n}] \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y'} \uplus \widetilde{z}}[\widetilde{x_1 \leftarrow x_n}], \text{ where } \widetilde{z} = \mathsf{lfv}(C').
\end{aligned}$$

The expression $\mathbb{N}$ can perform the reduction:

$$\mathbb{N} = \mathtt{fail}^{\widetilde{y}}\ B \longrightarrow_{[\mathtt{R:Cons1}]} \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y} \uplus \widetilde{z}}, \text{ where } \widetilde{z} = \mathsf{mlfv}(C)$$

Thus, $\mathbb{L} = (\!|\mathbb{N}'|\!)^{\circ}$ and so the result follows.

c. Suppose that $(\!|N'|\!)^{\bullet} \longrightarrow (\!|N''|\!)^{\bullet}$. This case follows from the induction hypothesis.

3. $\mathbb{N} = N\langle\langle B/x \rangle\rangle$:
Suppose $\mathsf{lfv}(N\langle\langle B/x\rangle\rangle) = \{x_1, \cdots, x_k\}$. Then,

$$
\begin{aligned}
(\!|\mathbb{N}|\!)^\circ = (\!|N\langle\langle B/x\rangle\rangle|\!)^\circ &= (\!|N\langle\langle B/x\rangle\rangle\langle\widetilde{x_1}/x_1\rangle\cdots\langle\widetilde{x_k}/x_k\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] \\
&= (\!|N'\langle\langle B'/x\rangle\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}]
\end{aligned} \tag{9}
$$

Let us consider the two possibilities of the encoding where we take $B = C \star U$:

a. Where $\#(x, M) = \mathsf{size}(B) = k$
Then we continue equation (9) as follows

$$
\begin{aligned}
(\!|\mathbb{N}|\!)^\circ &= (\!|N'\langle\langle B'/x\rangle\rangle|\!)^\bullet[[\widetilde{x_1 \leftarrow x_k}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} (\!|N'\langle y_1, \cdots, y_n/x\rangle|\!)^\bullet \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(n)/y_n|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] \\
&= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} (\!|N''|\!)^\bullet \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(n)/y_n|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}]
\end{aligned} \tag{10}
$$

There are five possible reductions that can take place, these being $[\texttt{RS:Fetch}^\ell]$, $[\texttt{RS:Fetch}^!]$, $[\texttt{RS:Fail}^!]$, $[\texttt{RS : Cons}_3]$ and when we apply the $[\texttt{RS : Cont}]$ rules

i. Suppose that $\mathsf{head}(N'') = y_1$ and for simplicity we assume C' has only one element N_1 then from (10) and buy letting $C' = \wr N_1' \wr$ we have

$$
\begin{aligned}
(\!|\mathbb{N}|\!)^\circ &= (\!|N''|\!)^\bullet \langle\!|\{\!\{N_1'\}\!\}/y_1|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow (\!|N^{''}|\!)^\bullet \{\!|\{\!\{N_1'\}\!\}/y_1|\!\} \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] = \mathbb{L}
\end{aligned}
$$

Also, $\mathbb{N} = N\langle\langle \wr N_1 \wr \star U/x\rangle\rangle \longrightarrow N\{\!|N_1/x|\!\}\langle\langle 1 \star U/x\rangle\rangle = \mathbb{N}'$. Then $\mathbb{L}' = (\!|\mathbb{N}'|\!)^\circ$ and the result follows.

ii. Suppose that $\mathsf{head}(N'') = x[i]$ and then from (10) we have

$$
\begin{aligned}
(\!|\mathbb{N}|\!)^\circ &= (\!|N''|\!)^\bullet \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(n)/y_n|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow (\!|N^{''}|\!)^\bullet \{\!|U_i/x[i]|\!\} \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(n)/y_n|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] = \mathbb{L}
\end{aligned}
$$

We also have that $\mathbb{N} = N\langle\langle C \star U/x\rangle\rangle \longrightarrow N\{\!|U_{ind}/x^!|\!\}\langle\langle C \star U/x\rangle\rangle = \mathbb{N}'$.
Then, $\mathbb{L}' = (\!|\mathbb{N}'|\!)^\circ$ and so the result follows.

iii. Suppose that $N'' = \texttt{fail}^{\widetilde{z'}}$ proceed similarly then from (10)

$$
\begin{aligned}
(\!|\mathbb{N}|\!)^\circ &= \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} \texttt{fail}^{\widetilde{z'}} \langle\!|C_i(1)/y_1|\!\rangle \cdots \langle\!|C_i(n)/y_n|\!\rangle \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow^* \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} \texttt{fail}^{(\widetilde{z'} \setminus y_1, \cdots, y_n) \uplus \widetilde{y}} \llbracket U/x \rrbracket [\widetilde{x_1 \leftarrow x_k}] \\
&\longrightarrow^* \sum_{C_i \in \mathsf{PER}((\!|C'|\!)^\bullet)} \texttt{fail}^{(\widetilde{z'} \setminus y_1, \cdots, y_n) \uplus \widetilde{y}} [\widetilde{x_1 \leftarrow x_k}] = \mathbb{L}'
\end{aligned}
$$

where $\widetilde{y} = \mathsf{lfv}(C_i(1)) \uplus \cdots \uplus \mathsf{lfv}(C_i(n))$. We also have that

$$\mathbb{N} = \mathtt{fail}^{\widetilde{z}}\langle\langle B/x\rangle\rangle \longrightarrow \mathtt{fail}^{(\widetilde{z}\setminus x)\uplus\widetilde{y}} = \mathbb{N}', \text{ where } \widetilde{y} = \mathsf{mfv}(B).$$

Then, $\mathbb{L}' = (\!|\mathbb{N}'|\!)^\circ$ and so the result follows.

iv. Suppose that $N'' \longrightarrow N'''$. This case follows by the induction hypothesis

b. Otherwise, we continue from equation (9), where $\#(x, M) \neq k$, as follows

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ &= (\!|N'\langle\langle B'/x\rangle\rangle|\!)^\bullet[\widetilde{x_1 \leftarrow x_k}] \\ &= (\!|N'\langle y_1.\cdots, y_k/x\rangle|\!)^\bullet[y_1.\cdots, y_k \leftarrow x]\langle\langle (\!|B'|\!)^\bullet/x\rangle\rangle[\widetilde{x_1 \leftarrow x_k}] \\ &= (\!|N''|\!)^\bullet[y_1.\cdots, y_k \leftarrow x]\langle\langle (\!|B'|\!)^\bullet/x\rangle\rangle[\widetilde{x_1 \leftarrow x_k}]\end{aligned}$$

We can perform the reduction

$$\begin{aligned}(\!|\mathbb{N}|\!)^\circ &= (\!|N''|\!)^\bullet[y_1.\cdots, y_k \leftarrow x]\langle\langle (\!|B'|\!)^\bullet/x\rangle\rangle[\widetilde{x_1 \leftarrow x_k}] \\ &\longrightarrow \sum_{C_i \in \mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}'}[\widetilde{x_1 \leftarrow x_k}], \text{ where } \widetilde{z}' = \mathsf{fv}(N'') \uplus \mathsf{fv}(C') = \mathbb{L}'\end{aligned}$$

We also have that

$$\mathbb{N} = N\langle\langle C/x\rangle\rangle \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}} = \mathbb{N}', \text{ where } \widetilde{z} = \mathsf{mlfv}(M) \uplus \mathsf{mlfv}(C).$$

Then, $\mathbb{L}' = (\!|\mathbb{N}'|\!)^\circ$ and so the result follows.

4. $\mathbb{N} = \mathbb{N}_1 + \mathbb{N}_2$:
Then this case holds by the induction hypothesis.

◀

E.3 Success Sensitiveness of $(\!| \cdot |\!)^\circ$

We now consider success sensitiveness, a property that complements (and relies on) operational completeness and soundness. For the purposes of the proof, we consider the extension of $u\lambda^{\lightning}_{\oplus}$ and $\widehat{u\lambda}^{\lightning}_{\oplus}$ with dedicated constructs and predicates that specify success.

▶ **Definition 58.** *We extend the syntax of terms for $u\lambda^{\lightning}_{\oplus}$ and $\widehat{u\lambda}^{\lightning}_{\oplus}$ with the same ✓ construct. In both cases, we assume ✓ is well formed. Also, we also define* $\mathsf{head}(✓) = ✓$ *and* $(\!|✓|\!)^\bullet = ✓$

An expression $\mathbb{M}$ has success, denoted $\mathbb{M} \Downarrow_{✓}$, when there is a sequence of reductions from $\mathbb{M}$ that leads to an expression that includes a summand that contains an occurrence of ✓ in head position.

▶ **Definition 59** (Success in $u\lambda^{\lightning}_{\oplus}$ and $\widehat{u\lambda}^{\lightning}_{\oplus}$). *In $u\lambda^{\lightning}_{\oplus}$ and $\widehat{u\lambda}^{\lightning}_{\oplus}$, we define $\mathbb{M} \Downarrow_{✓}$ if and only if there exist $M_1, \cdots, M_k$ such that $\mathbb{M} \longrightarrow^* M_1 + \cdots + M_k$ and* $\mathsf{head}(M_j') = ✓$, *for some $j \in \{1, \ldots, k\}$ and term M_j' such that $M_j \equiv_\lambda M_j'$.*

▶ **Notation 7.** *We use the notation* $\mathsf{head}_{\sum}(\mathbb{M})$ *to be that $\forall M_i, M_j \in \mathbb{M}$ we have that* $\mathsf{head}(M_i) = \mathsf{head}(M_j)$ *hence we say that* $\mathsf{head}_{\sum}(\mathbb{M}) = \mathsf{head}(M_i)$ *for some $M_i \in \mathbb{M}$*

▶ **Proposition 60** (Preservation of Head term). *The head of a term is preserved when applying the encoding* $(\!|\text{-}|\!)^{\bullet}$*. That is to say:*

$$\forall M \in u\lambda^{\lightning}_{\oplus} \quad \mathsf{head}(M) = \checkmark \iff \mathsf{head}_{\sum}((\!|M|\!)^{\circ}) = \checkmark$$

Proof. By induction on the structure of M. We only need to consider terms of the following form.

1. When $M = \checkmark$ the case is immediate.
2. When $M = NB$ with $\mathsf{fv}(NB) = \{x_1, \cdots, x_k\}$ and $\#(x_i, M) = j_i$ we have that:

$$\begin{aligned}\mathsf{head}_{\sum}((\!|NB|\!)^{\circ}) &= \mathsf{head}_{\sum}((\!|NB\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_k}/x_k\rangle|\!)^{\bullet}[\widetilde{x_1} \leftarrow x_1] \cdots [\widetilde{x_k} \leftarrow x_k]) \\ &= \mathsf{head}_{\sum}((\!|NB|\!)^{\bullet}) = \mathsf{head}_{\sum}((\!|N|\!)^{\bullet})\end{aligned}$$

   and $\mathsf{head}(NB) = \mathsf{head}(N)$, by the IH we have $\mathsf{head}(N) = \checkmark \iff \mathsf{head}_{\sum}((\!|N|\!)^{\bullet}) = \checkmark$.
3. When $M = N\langle\!\langle C \star U/x\rangle\!\rangle$, we must have that $\#(x, M) = \mathsf{size}(C)$ for the head of this term to be $\checkmark$. Let $\mathsf{fv}(N\langle\!\langle C \star U/x\rangle\!\rangle) = \{x_1, \cdots, x_k\}$ and $\#(x_i, M) = j_i$. We have that:

$$\begin{aligned}\mathsf{head}_{\sum}((\!|N\langle\!\langle C \star U/x\rangle\!\rangle|\!)^{\circ}) &= \mathsf{head}_{\sum}((\!|N\langle\!\langle C \star U/x\rangle\!\rangle\langle \widetilde{x_1}/x_1\rangle \cdots \langle \widetilde{x_k}/x_k\rangle|\!)^{\bullet}[\widetilde{x_1 \leftarrow x_k}]) \\ &= \mathsf{head}_{\sum}((\!|N\langle\!\langle C \star U/x\rangle\!\rangle|\!)^{\bullet}) \\ &= \mathsf{head}_{\sum}(\sum_{C_i \in \mathsf{PER}((\!|C|\!)^{\bullet})} (\!|N\langle \widetilde{x}/x\rangle|\!)^{\bullet}\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket) \\ &= \mathsf{head}_{\sum}((\!|N\langle \widetilde{x}/x\rangle|\!)^{\bullet}\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \llbracket U/x \rrbracket) \\ &= \mathsf{head}_{\sum}((\!|N\langle \widetilde{x}/x\rangle|\!)^{\bullet})\end{aligned}$$

   and $\mathsf{head}(N\langle\!\langle B/x\rangle\!\rangle) = \mathsf{head}(N)$, by the IH $\mathsf{head}(N) = \checkmark \iff \mathsf{head}_{\sum}((\!|N|\!)^{\bullet}) = \checkmark$.

◀

▶ **Theorem 61** (Success Sensitivity). *Let* $\mathbb{M}$ *be a well-formed expression. We have* $\mathbb{M} \Downarrow_{\checkmark}$ *if and only if* $(\!|\mathbb{M}|\!)^{\circ} \Downarrow_{\checkmark}$.

Proof. By induction on the structure of expressions $u\lambda^{\lightning}_{\oplus}$ and $u\widehat{\lambda}^{\lightning}_{\oplus}$.

1. Suppose that $\mathbb{M} \Downarrow_{\checkmark}$. We will prove that $(\!|\mathbb{M}|\!)^{\circ} \Downarrow_{\checkmark}$.
   By operational completeness (Theorem 56) we have that if $\mathbb{M} \longrightarrow_{[\mathtt{R}]} \mathbb{M}'$ then

   a. If $[\mathtt{R}] = [\mathtt{R} : \mathtt{Beta}]$ then $(\!|\mathbb{M}|\!)^{\circ} \longrightarrow^{\leq 2} (\!|\mathbb{M}'|\!)^{\circ}$;
   b. If $[\mathtt{R}] = [\mathtt{R} : \mathtt{Fetch}]$ then $(\!|\mathbb{M}|\!)^{\circ} \longrightarrow^{+} (\!|\mathbb{M}''|\!)^{\circ}$, for some $\mathbb{M}''$ such that $\mathbb{M}' \equiv_{\lambda} \mathbb{M}''$.
   c. If $[\mathtt{R}] \neq [\mathtt{R} : \mathtt{Beta}]$ and $[\mathtt{R}] \neq [\mathtt{R} : \mathtt{Fetch}]$ then $(\!|\mathbb{M}|\!)^{\circ} \longrightarrow (\!|\mathbb{M}'|\!)^{\circ}$;

   Notice that neither our reduction rules (in Def. 13), or our congruence $\equiv_{\lambda}$ (in Fig. 23), or our encoding ($(\!|\checkmark|\!)^{\circ} = \checkmark$) create or destroy a $\checkmark$ occurring in the head of term. By Proposition 60 the encoding preserves the head of a term being $\checkmark$. The encoding acts homomorphically over sums, therefore, if a $\checkmark$ appears as the head of a term in a sum, it will stay in the encoded sum. We can iterate the operational completeness lemma and obtain the result.

2. Suppose that $(\!|\mathbb{M}|\!)^\circ \Downarrow_{\checkmark}$. We will prove that $\mathbb{M} \Downarrow_{\checkmark}$.
By operational soundness (Theorem 57) we have that if $(\!|\mathbb{M}|\!)^\circ \longrightarrow \mathbb{L}$ then there exist $\mathbb{M}'$ such that $\mathbb{M} \longrightarrow_{[\mathtt{R}]} \mathbb{M}'$ and

   a. If $[\mathtt{R}] = [\mathtt{R} : \mathtt{Beta}]$ then $\mathbb{L} \longrightarrow^{\leq 1} (\!|\mathbb{M}'|\!)^\circ$;
   b. If $[\mathtt{R}] \neq [\mathtt{R} : \mathtt{Beta}]$ then $\mathbb{L} \longrightarrow^{*} (\!|\mathbb{M}''|\!)^\circ$, for $\mathbb{M}''$ such that $\mathbb{M}' \equiv_\lambda \mathbb{M}''$.

   Since $(\!|\mathbb{M}|\!)^\circ \longrightarrow^* M_1 + \ldots + M_k$, and $\mathsf{head}(M'_j) = \checkmark$, for some j and M'_j, s.t. $M_j \equiv_\lambda M'_j$. Notice that if $(\!|\mathbb{M}|\!)^\circ$ is itself a term headed with $\checkmark$, say $\mathsf{head}((\!|\mathbb{M}|\!)^\circ) = \checkmark$, then $\mathbb{M}$ is itself headed with $\checkmark$, from Proposition 60. In the case $(\!|\mathbb{M}|\!)^\circ = M_1 + \ldots + M_k$, $k \geq 2$, and $\checkmark$ occurs in the head of an M_j, the reasoning is similar. $\mathbb{M}$ has one of the forms:
   a. $\mathbb{M} = N_1$, then N_1 must contain the subterm $M\langle\!\langle C \star U/x\rangle\!\rangle$ and $\mathsf{size}(C) = \#(x, M)$.
   The encoding of $\mathbb{M}$ is
   $(\!|M\langle\!\langle C \star U/x\rangle\!\rangle|\!)^\circ = \sum_{C_i \in \mathsf{PER}((\!|C|\!)^\bullet)} (\!|M\langle \widetilde{x}/x\rangle|\!)^\bullet \langle\!| C_i(1)/x_i |\!\rangle \ldots \langle\!| C_i(k)/x_i |\!\rangle [\![U/x]\!]$. We can apply Proposition 60 and the result follows.
   b. $\mathbb{M} = N_1 + \ldots + N_l$ for $l \geq 2$.
   This reasoning is similar and uses the fact that the encoding distributes homomorphically over sums.
   In the case where $(\!|\mathbb{M}|\!)^\circ \longrightarrow^+ M_1 + \ldots + M_k$, and $\mathsf{head}(M'_j) = \checkmark$, for some j and M'_j, such that $M_j \equiv_\lambda M'_j$, the reasoning is similar to the previous, since our reduction rules do not introduce/eliminate $\checkmark$ occurring in the head of terms.

◀

F Appendix to Subsection 4.4

F.1 Type Preservation

▶ **Lemma 62.** *$[\![\sigma^j]\!]_{(\tau_1,m)} = [\![\sigma^k]\!]_{(\tau_2,n)}$ and $[\![(\sigma^j,\eta)]\!]_{(\tau_1,m)} = [\![(\sigma^k,\eta)]\!]_{(\tau_2,n)}$ hold, provided that τ_1, τ_2, n and m are as follows:*

1. *If $j > k$ then take τ_1 to be an arbitrary type, $m = 0$, take τ_2 to be σ and $n = j - k$.*
2. *If $j < k$ then take τ_1 to be σ, $m = k - j$, take τ_2 to be an arbitrary type and $n = 0$.*
3. *Otherwise, if $j = k$ then take $m = n = 0$. In this case, $\tau_1.\tau_2$ are unimportant.*

Proof. We shall prove the case of (1) for the first equality, and the case for the second equality and of (2) are analogous. The case of (3) follows by the encoding on types in Def. 28.

Hence take $j, k, \tau_1, \tau_2, m, n$ satisfying the conditions in (1): $j > k$, τ_1 to be an arbitrary type, $m = 0$, $\tau_2 = \sigma$ and $n = j - k$. We want to show that $[\![\sigma^j]\!]_{(\tau_1,0)} = [\![\sigma^k]\!]_{(\sigma,n)}$. In fact,

$$
\begin{aligned}
[\![\sigma^k]\!]_{(\sigma,n)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma^{k-1}]\!]_{(\sigma,n)})))) \\
[\![\sigma^{k-1}]\!]_{(\sigma,n)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma^{k-2}]\!]_{(\sigma,n)})))) \\
&\vdots \\
[\![\sigma^1]\!]_{(\sigma,n)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\omega]\!]_{(\sigma,n)}))))
\end{aligned}
$$

and

$$
\begin{aligned}
[\![\sigma^j]\!]_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma^{j-1}]\!]_{(\tau_1,0)})))) \\
[\![\sigma^{j-1}]\!]_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma^{j-2}]\!]_{(\tau_1,0)})))) \\
&\vdots \\
[\![\sigma^{j-k+1}]\!]_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma^{j-k}]\!]_{(\tau_1,0)}))))
\end{aligned}
$$

Notice that $n = j - k$, hence we wish to show that $\llbracket \sigma^n \rrbracket_{(\tau_1,0)} = \llbracket \omega \rrbracket_{(\sigma,n)}$. Finally,

$$\begin{aligned}
\llbracket \omega \rrbracket_{(\sigma,n)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \omega \rrbracket_{(\sigma,n-1)})))) \\
\llbracket \omega \rrbracket_{(\sigma,n-1)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \omega \rrbracket_{(\sigma,n-2)})))) \\
&\vdots \\
\llbracket \omega \rrbracket_{(\sigma,1)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \omega \rrbracket_{(\sigma,0)})))) \\
\llbracket \omega \rrbracket_{(\sigma,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&\mathbf{1})
\end{aligned}$$

and

$$\begin{aligned}
\llbracket \sigma^n \rrbracket_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \sigma^{n-1} \rrbracket_{(\tau_1,0)})))) \\
\llbracket \sigma^{n-1} \rrbracket_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \sigma^{n-2} \rrbracket_{(\tau_1,0)})))) \\
&\vdots \\
\llbracket \sigma^1 \rrbracket_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus\llbracket \sigma \rrbracket) \otimes (\llbracket \omega \rrbracket_{(\tau_1,0)})))) \\
\llbracket \omega \rrbracket_{(\tau_1,0)} &= \oplus((\&\mathbf{1}) \parr (\oplus\&\mathbf{1})
\end{aligned}$$

◀

▶ **Lemma 63.** *If $\eta \propto \epsilon$ Then*
1. *If $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket; \llbracket \Theta \rrbracket, x^! : \llbracket \eta \rrbracket$ then $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket; \llbracket \Theta \rrbracket, x^! : \llbracket \epsilon \rrbracket$.*
2. *If $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket, u : \llbracket (\sigma^j, \eta) \to \tau \rrbracket; \llbracket \Theta \rrbracket$ then $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket, u : \llbracket (\sigma^j, \epsilon) \to \tau \rrbracket; \llbracket \Theta \rrbracket$.*

Proof. 1. We consider the first case where if $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket; \llbracket \Theta \rrbracket, x^! : \llbracket \eta \rrbracket$ then $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket; \llbracket \Theta \rrbracket, x^! : \llbracket \epsilon \rrbracket$ and by Def. 28, $\llbracket \eta \rrbracket = \&_{\eta_i \in \eta}\{\mathtt{l}_i; \llbracket \eta_i \rrbracket\}$. We now proceed by induction on the structure of M:

a. $M = x$.

By Fig. 8, $\llbracket x \rrbracket_u = x.\overline{\mathtt{some}}; [x \leftrightarrow u]$. We have the following derivation:

$$\dfrac{\dfrac{}{[x \leftrightarrow u] \vdash x : \overline{A}, u : A; \llbracket \Theta \rrbracket, x^! : \llbracket \eta \rrbracket}\,[(\mathtt{Tid})]}{x.\overline{\mathtt{some}}; [x \leftrightarrow u] \vdash x : \&\overline{A}, u : A; \llbracket \Theta \rrbracket, x^! : \llbracket \eta \rrbracket}\,[\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}})]$$

For some type A. Notice the derivation is independent of $x^! : \llbracket \eta \rrbracket$, hence holds when $M = x$. Note that we do not consider $M = y$ where $y \neq x$, this is due to the case being trivial due to the typing of y being independent on x.

b. $M = x[ind]$.

By Fig. 8, $\llbracket x[ind] \rrbracket_u = \overline{x^!}?(x_i).x_i.l_{ind}; [x_i \leftrightarrow u]$. We have the following derivation:

$$\dfrac{\dfrac{\dfrac{}{[x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \overline{\llbracket \eta_{ind} \rrbracket}; x^! : \&_{\eta_i \in \eta}\{\mathtt{l}_i; \llbracket \eta_i \rrbracket\}, \llbracket \Theta \rrbracket}\,[(\mathtt{Tid})]}{x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}; x^! : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}, \llbracket \Theta \rrbracket}\,[\mathtt{T}\oplus_{\mathtt{i}}]}{\overline{x^!}?(x_i).x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket; x^! : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}, \llbracket \Theta \rrbracket}\,[\mathtt{Tcopy}]$$

On the other hand we have derivation:

$$\dfrac{\dfrac{\dfrac{}{[x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \overline{\llbracket \epsilon_{ind} \rrbracket}; x^! : \&_{\epsilon_i \in \epsilon}\{\mathtt{l}_i; \llbracket \epsilon_i \rrbracket\}, \llbracket \Theta \rrbracket}\,[(\mathtt{Tid})]}{x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \oplus_{\epsilon_i \in \epsilon}\{\mathtt{l}_i; \overline{\llbracket \epsilon_i \rrbracket}\}; x^! : \oplus_{\epsilon_i \in \epsilon}\{\mathtt{l}_i; \overline{\llbracket \epsilon_i \rrbracket}\}, \llbracket \Theta \rrbracket}\,[\mathtt{T}\oplus_{\mathtt{i}}]}{\overline{x^!}?(x_i).x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket; x^! : \oplus_{\epsilon_i \in \epsilon}\{\mathtt{l}_i; \overline{\llbracket \epsilon_i \rrbracket}\}, \llbracket \Theta \rrbracket}\,[\mathtt{Tcopy}]$$

By $\eta \propto \epsilon$ we have that $\epsilon_{ind} = \eta_{ind}$. Similarly for the case of $M = y[ind]$ with $y \neq x$ we use the argument that the typing of y is independent on x.

c. $M = M'[\widetilde{y} \leftarrow y]$.
If $y = x$ the case proceeds similarly to (1a) otherwise we proceed by induction on M'.

d. $M = \lambda x.(M'[\widetilde{x} \leftarrow x])$.
From Def. 26 it follows that
$[\![\lambda x.M'[\widetilde{x} \leftarrow x]]\!]_u = u.\overline{\mathtt{some}}; u(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u$.
We give the final derivation in parts. The first part we name Π_1 derived by:

$$
\dfrac{\dfrac{\dfrac{\dfrac{[\![M'[\widetilde{x} \leftarrow x]]\!]_u \vdash u : [\![\tau]\!], [\![\Gamma']\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!], x^! : \overline{[\![\eta]\!]}}{x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \vdash x{:}\bot, u : [\![\tau]\!], [\![\Gamma']\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!], x^! : \overline{[\![\eta]\!]}}\,[\mathtt{T}\bot]}{x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \vdash x{:}\bot, u : [\![\tau]\!], [\![\Gamma']\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}, x^! : ?\overline{[\![\eta]\!]}; [\![\Theta]\!]}\,[\mathtt{T}?]}{x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \vdash x : (?\overline{[\![\eta]\!]}) \parr (\bot), u : [\![\tau]\!], [\![\Gamma']\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!]}\,[\mathtt{T}\parr]}{x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \vdash x : \overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot)), u : [\![\tau]\!], [\![\Gamma']\!]; [\![\Theta]\!]}\,[\mathtt{T}\parr]
$$

We take $P = x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u$ and continue the derivation:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Pi_1}{\vdots}}{\dfrac{P \vdash x : \overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot)), u : [\![\tau]\!], [\![\Gamma']\!]; [\![\Theta]\!]}{x.\overline{\mathtt{some}}; P \vdash x : \&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))), u : [\![\tau]\!], [\![\Gamma']\!]; [\![\Theta]\!]}\,[\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}]}}{u(x).x.\overline{\mathtt{some}}; P \vdash u : \&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!], [\![\Gamma']\!]; [\![\Theta]\!]}\,[\mathtt{T}\parr]}{u.\overline{\mathtt{some}}; u(x).x.\overline{\mathtt{some}}; P \vdash u : \&(\&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!]), [\![\Gamma']\!]; [\![\Theta]\!]}\,[\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}]
$$

By Definition 28 we have that $[\![(\sigma^k, \eta) \to \tau]\!] = \&(\&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!])$.
In this case we must have that the variable names for x from our hypothesis and x from M must be distinct.

e. $M = (M'\ B)$, or $M = (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle$, or $M = M'\llbracket U/x \rrbracket$.
The proof follows similarly to that of (1b).

f. $M = M'\langle\!| N/x |\!\rangle$
Case follows by that of (1a) and applying induction hypothesis on $[\![M']\!]_u$.

g. When $M = \mathtt{fail}^{\widetilde{x}}$ Case follows by that of (1a).

2. If $[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![(\sigma^j, \eta) \to \tau]\!]; [\![\Theta]\!]$ then $[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![(\sigma^j, \epsilon) \to \tau]\!]; [\![\Theta]\!]$ follows from previous case along a similar argument.

◀

▶ **Theorem 64** (Type Preservation for $[\![\cdot]\!]_u$). *Let B and $\mathbb{M}$ be a bag and an expression in $u\widehat{\lambda}^{\lightning}_{\oplus}$, respectively.*

1. *If $\Theta; \Gamma \models B : (\sigma^k, \eta)$ then $[\![B]\!]_u \models [\![\Gamma]\!], u : [\![(\sigma^k, \eta)]\!]_{(\sigma,i)}; [\![\Theta]\!]$.*
2. *If $\Theta; \Gamma \models \mathbb{M} : \tau$ then $[\![\mathbb{M}]\!]_u \models [\![\Gamma]\!], u : [\![\tau]\!]; [\![\Theta]\!]$.*

Proof. The proof is by mutual induction on the typing derivation of B and $\mathbb{M}$, with an analysis for the last rule applied. Recall that the encoding of types ($[\![-]\!]$) has been given in Def. 28.

1. We have the following derivation where we take $B = C \star U$:

$$
[\texttt{FS:bag}]\ \frac{\Theta;\Gamma \models C : \sigma^k \qquad \Theta;\cdot \models U : \eta}{\Theta;\Gamma \models C \star U : (\sigma^k, \eta)}
$$

Our encoding gives: $\llbracket C \star U \rrbracket_u = x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^\ell).(\llbracket C \rrbracket_{x^\ell} \mid \overline{x}(x^!).(!x^!(x_i).\llbracket U \rrbracket_{x_i} \mid x.\overline{\mathtt{close}}))$.
In addition, the encoding of (σ^k, η) is:

$$
\llbracket (\sigma^k, \eta) \rrbracket_{(\sigma,i)} = \oplus((\llbracket \sigma^k \rrbracket_{(\sigma,i)}) \otimes ((!\llbracket \eta \rrbracket) \otimes (\mathbf{1}))) \quad \text{(for some } i \geq 0 \text{ and strict type } \sigma)
$$

And one can build the following type derivation (rules from Fig. 6):

$$
[\mathtt{T}\oplus^{\mathtt{x}}_{\widetilde{\mathtt{w}}}]\ \dfrac{[\mathtt{T}\otimes]\ \dfrac{\llbracket C \rrbracket_{x^\ell} \vdash \llbracket \Gamma \rrbracket, x^\ell : \llbracket \sigma^k \rrbracket_{(\sigma,i)}; \llbracket \Theta \rrbracket \qquad [\mathtt{T}\otimes]\ \dfrac{[\mathtt{T}!]\ \dfrac{\llbracket U \rrbracket_{x_i} \vdash x_i : \llbracket \eta \rrbracket; \llbracket \Theta \rrbracket}{!x^!(x_i).\llbracket U \rrbracket_{x_i} \vdash x^! : !\llbracket \eta \rrbracket; \llbracket \Theta \rrbracket} \qquad [\mathtt{T}\mathbf{1}]\ \dfrac{}{x.\overline{\mathtt{close}} \vdash x : \mathbf{1}; \llbracket \Theta \rrbracket}}{\overline{x}(x^!).(!x^!.(x_i).\llbracket U \rrbracket_{x_i} \mid x.\overline{\mathtt{close}} \vdash x : (!\llbracket \eta \rrbracket) \otimes (\mathbf{1}); \llbracket \Theta \rrbracket}}{\overline{x}(x^\ell).(\llbracket C \rrbracket_{x^\ell} \mid \overline{x}(x^!).(!x^!.(x_i).\llbracket U \rrbracket_{x_i} \mid x.\overline{\mathtt{close}})) \vdash \llbracket \Gamma \rrbracket, x : (\llbracket \sigma^k \rrbracket_{(\sigma,i)}) \otimes ((!\llbracket \eta \rrbracket) \otimes (\mathbf{1})); \llbracket \Theta \rrbracket}}{x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^\ell).(\llbracket C \rrbracket_{x^\ell} \mid \overline{x}(x^!).(!x^!.(x_i).\llbracket U \rrbracket_{x_i} \mid x.\overline{\mathtt{close}})) \vdash \llbracket \Gamma \rrbracket, x{:}\llbracket (\sigma^k, \eta) \rrbracket_{(\sigma,i)})}
$$

Hence true provided both $\llbracket C \rrbracket_{x^\ell} \vdash \llbracket \Gamma \rrbracket, x^\ell : \llbracket \sigma^k \rrbracket_{(\sigma,i)}; \llbracket \Theta \rrbracket$ and $\llbracket U \rrbracket_{x_i} \vdash x_i : \llbracket \eta \rrbracket; \llbracket \Theta \rrbracket$ hold.
Let us consider the two cases:

a. For $\llbracket C \rrbracket_{x^\ell} \vdash \llbracket \Gamma \rrbracket, x^\ell : \llbracket \sigma^k \rrbracket_{(\sigma,i)}; \llbracket \Theta \rrbracket$ to hold we must consider two cases on the shape of C:

i. When $C = \mathtt{1}$ we may type bags with the $[\texttt{FS:1}^\ell]$ rule.
That is,

$$
[\texttt{FS:1}^\ell]\ \frac{}{\Theta; \text{-} \models \mathtt{1} : \omega}
$$

Our encoding gives: $\llbracket \mathtt{1} \rrbracket_{x^\ell} = x^\ell.\mathtt{some}_\emptyset; x^\ell(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}})$.
and the encoding of ω can be either:

A. $\llbracket \omega \rrbracket_{(\sigma,0)} = \overline{\&((\oplus\bot) \otimes (\& \oplus \bot))}$; or

B. $\llbracket \omega \rrbracket_{(\sigma,i)} = \overline{\&((\oplus\bot) \otimes (\& \oplus ((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i-1)}}))))}$

And one can build the following type derivation (rules from Fig. 6):

$$
[\mathtt{T}\oplus^{\mathtt{x}}_{\widetilde{\mathtt{w}}}]\ \dfrac{[\mathtt{T}\parr]\ \dfrac{[\mathtt{T}\ \|]\ \dfrac{[\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}]\ \dfrac{[\mathtt{T}\mathbf{1}]\ \dfrac{}{y_n.\overline{\mathtt{close}} \vdash y_n : \mathbf{1}; \llbracket \Theta \rrbracket}}{y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \vdash y_n : \&\mathbf{1}; \llbracket \Theta \rrbracket} \qquad [\mathtt{T}\oplus^{\mathtt{x}}_{\widetilde{\mathtt{w}}}]\ \dfrac{[\mathtt{T}\&^{\mathtt{x}}]\ \dfrac{}{x^\ell.\overline{\mathtt{none}} \vdash x^\ell : \&A; \llbracket \Theta \rrbracket}}{x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}} \vdash x^\ell{:} \oplus \&A; \llbracket \Theta \rrbracket}}{(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \vdash y_n : \&\mathbf{1}, x^\ell{:} \oplus \&A; \llbracket \Theta \rrbracket}}{x^\ell(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \vdash x^\ell : (\&\mathbf{1}) \parr (\oplus\&A); \llbracket \Theta \rrbracket}}{x^\ell.\mathtt{some}_\emptyset; x^\ell(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \vdash x^\ell{:} \oplus ((\&\mathbf{1}) \parr (\oplus\&A)); \llbracket \Theta \rrbracket}
$$

Since A is arbitrary, we can take $A = \mathtt{1}$ for $\llbracket \omega \rrbracket_{(\sigma,0)}$ and $A = \overline{((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i-1)}}))}$ for $\llbracket \omega \rrbracket_{(\sigma,i)}$, in both cases, the result follows.

ii. When $C = \wr M \int \cdot C'$ we may type bags with the $[\texttt{FS:bag}^\ell]$ rule.

$$
[\texttt{FS:bag}^\ell]\ \frac{\Theta;\Gamma' \models M : \sigma \qquad \Theta;\Delta \models C' : \sigma^k}{\Theta;\Gamma',\Delta \models \wr M \int \cdot C' : \sigma^k}
$$

Where $\Gamma = \Gamma', \Delta$. To simplify the proof, we will consider $k = 3$.
By IH we have

$$[\![M]\!]_{x_i} \vdash [\![\Gamma']\!], x_i : [\![\sigma]\!]; [\![\Theta]\!] \qquad [\![C']\!]_{x^\ell} \qquad \vdash [\![\Delta]\!], x^\ell : [\![\sigma \wedge \sigma]\!]_{(\tau,j)}; [\![\Theta]\!]$$

By Def. 26,

$$\begin{aligned}[\![\wr M \int \cdot C']\!]_{x^\ell} = & x^\ell.\mathtt{some}_{\mathsf{lfv}(\wr M \int \cdot C)}; x^\ell(y_i).x^\ell.\mathtt{some}_{y_i,\mathsf{lfv}(\wr M \int \cdot C)}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_i). \\ & (x_i.\mathtt{some}_{\mathsf{lfv}(M)}; [\![M]\!]_{x_i} \mid [\![C']\!]_{x^\ell} \mid y_i.\overline{\mathtt{none}})\end{aligned} \tag{11}$$

Let Π_1 be the derivation:

$$\dfrac{\dfrac{[\![M]\!]_{x_i} \vdash [\![\Gamma']\!], x_i : [\![\sigma]\!]; [\![\Theta]\!]}{x_i.\mathtt{some}_{\mathsf{lfv}(M)}; [\![M]\!]_{x_i} \vdash [\![\Gamma']\!], x_i : \oplus [\![\sigma]\!]; [\![\Theta]\!]}[\mathtt{T}\oplus^{\mathtt{x}}_{\mathtt{w}}] \quad \dfrac{}{y_i.\overline{\mathtt{none}} \vdash y_i : \&\mathbf{1}; [\![\Theta]\!]}[\mathtt{T}\&^{\mathtt{x}}]}{\underbrace{x_i.\mathtt{some}_{\mathsf{lfv}(M)}; [\![M]\!]_{x_i} \mid y_i.\overline{\mathtt{none}}}_{P_1} \vdash [\![\Gamma']\!], x_i : \oplus[\![\sigma]\!], y_i : \&\mathbf{1}; [\![\Theta]\!]}[\mathtt{T} \mid]$$

Let $P_1 = (x_i.\mathtt{some}_{\mathsf{lfv}(M)}; [\![M]\!]_{x_i} \mid y_i.\overline{\mathtt{none}})$, in the the derivation Π_2 below:

$$\dfrac{\dfrac{\Pi_1 \qquad [\![C']\!]_{x^\ell} \vdash [\![\Delta]\!], x^\ell : [\![\sigma \wedge \sigma]\!]_{(\tau,j)}; [\![\Theta]\!]}{\overline{x^\ell}(x_i).(P_1 \mid [\![C']\!]_{x^\ell}) \vdash [\![\Gamma']\!], [\![\Delta]\!], y_i : \&\mathbf{1}, x^\ell : (\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)}); [\![\Theta]\!]}[\mathtt{T}\otimes]}{\underbrace{x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_i).(P_1 \mid [\![C']\!]_{x^\ell})}_{P_2} \vdash [\![\Gamma']\!], [\![\Delta]\!], y_i : \&\mathbf{1}, x^\ell : \&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)})); [\![\Theta]\!]}[\mathtt{T}\&^{\mathtt{x}}_{\mathtt{d}}]$$

Let $P_2 = (x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_i).(P_1 \mid [\![A]\!]_{x^\ell}))$ in the derivation below:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_2 \\ \vdots \\ P_2 \vdash [\![\Gamma]\!], y_i : \&\mathbf{1}, x^\ell : \&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)})); [\![\Theta]\!]\end{array}}{x^\ell.\mathtt{some}_{y_i,\mathsf{lfv}(\wr M \int \cdot C')}; P_2 \vdash [\![\Gamma]\!], y_i : \&\mathbf{1}, x^\ell : \oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)})); [\![\Theta]\!]}[\mathtt{T}\oplus^{\mathtt{x}}_{\mathtt{w}}]}{x^\ell(y_i).x^\ell.\mathtt{some}_{y_i,\mathsf{lfv}(\wr M \int \cdot C')}; P_2 \vdash [\![\Gamma]\!], x^\ell : (\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)}))); [\![\Theta]\!]}[\mathtt{T}\parr]}{[\![\wr M \int \cdot C']\!]_{x^\ell} \vdash [\![\Gamma]\!], x^\ell : \oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)})))); [\![\Theta]\!]}[\mathtt{T}\oplus^{\mathtt{x}}_{\mathtt{w}}]$$

From Definitions 21 (duality) and 28, we infer:

$$\oplus((\&\mathbf{1}) \parr (\oplus\&((\oplus[\![\sigma]\!]) \otimes ([\![\sigma \wedge \sigma]\!]_{(\tau,j)})))) = [\![\sigma \wedge \sigma \wedge \sigma]\!]_{(\tau,j)}$$

Therefore, $[\![\wr M \int \cdot C']\!]_{x^\ell} \vdash [\![\Gamma]\!], x^\ell : [\![\sigma \wedge \sigma \wedge \sigma]\!]_{(\tau,j)}$ and the result follows.

b. For $[\![U]\!]_{x_i} \vdash x_i : [\![\eta]\!]; [\![\Theta]\!]$ we consider U to be a binary concatenation of 2 components, one being an empty unrestricted bag and the other being $\wr M \int^!$. Hence we take $U = \mathtt{1}^! \diamond \wr M \int^!$ with $\eta = \sigma_1 \diamond \sigma_2$, $[\![\eta_i]\!] = \&\{\mathtt{l}_1; [\![\sigma_1]\!], \mathtt{l}_2; [\![\sigma_2]\!]\}$ by Def. 28 and finally by Def. 26 we have $[\![U]\!]_{x_i} = x_i.\mathtt{case}\{\mathtt{l}_1 : [\![\mathtt{1}^!]\!]_{x_i}, \mathtt{l}_2 : [\![\wr M \int^!]\!]_{x_i}\}$, $[\![\mathtt{1}^!]\!]_{x_i} = x_i.\overline{\mathtt{none}}$ and $[\![\wr M \int^!]\!]_{x_i} = [\![M]\!]_{x_i}$, we can conclude $[\![U]\!]_{x_i} = x_i.\mathtt{case}\{\mathtt{l}_1 : x_i.\overline{\mathtt{none}}, \mathtt{l}_2 : [\![M]\!]_{x_i}\}$.
Hence we have:

$$\dfrac{\dfrac{}{\Theta; - \models \mathtt{1}^! : \sigma_1}[\mathtt{FS{:}bag}^!] \quad \dfrac{\Theta; \cdot \models M : \sigma_2}{\Theta; \cdot \models \wr M \int^! : \sigma_2}[\mathtt{FS{:}bag}^!]}{\Theta; \cdot \models \mathtt{1}^! \diamond \wr M \int^! : \sigma_1 \diamond \sigma_2}[\mathtt{FS{:}\diamond-bag}^!]$$

By the induction hypothesis we have that $\Theta; \cdot \models M : \sigma$ implies $[\![M]\!]_{x_i} \models x_i : [\![\sigma]\!]; [\![\Theta]\!]$

$$
[\mathtt{T\&}]\ \dfrac{[\mathtt{T\&^{x}}]\ \dfrac{}{x_i.\overline{\mathtt{none}} \vdash x_i : \llbracket \sigma_1 \rrbracket; \llbracket \Theta \rrbracket} \qquad \llbracket M \rrbracket_{x_i} \vdash x_i : \llbracket \sigma_2 \rrbracket; \llbracket \Theta \rrbracket}{x_i.\mathtt{case}\{\mathtt{l}_1 : x_i.\mathtt{none}, \mathtt{l}_2 : \llbracket M \rrbracket_{x_i}\} \vdash x_i : \&\{\mathtt{l}_1; \llbracket \sigma_1 \rrbracket, \mathtt{l}_2; \llbracket \sigma_2 \rrbracket\}; \llbracket \Theta \rrbracket}
$$

Therefore, $x_i.\mathtt{case}\{\mathtt{l}_1 : x_i.\mathtt{none}, \mathtt{l}_2 : \llbracket M \rrbracket_{x_i}\} \vdash x_i : \&\{\mathtt{l}_1; \llbracket \sigma_1 \rrbracket, \mathtt{l}_2; \llbracket \sigma_2 \rrbracket\}; \llbracket \Theta \rrbracket$ and the result follows.

2. The proof of type preservation for expressions, relies on the analysis of twelve cases:

   a. **Rule** [$\mathtt{FS{:}var}^{\ell}$]**:** Then we have the following derivation:

$$
[\mathtt{FS{:}var}^{\ell}]\ \dfrac{}{\Theta; x : \tau \models x : \tau}
$$

By Def. 28, $\llbracket x : \tau \rrbracket = x : \&\overline{\llbracket \tau \rrbracket}$, and by Fig. 8, $\llbracket x \rrbracket_u = x.\overline{\mathtt{some}}; [x \leftrightarrow u]$. The thesis holds thanks to the following derivation:

$$
[\mathtt{T\&^{x}_{d}})]\ \dfrac{[(\mathtt{Tid})]\ \dfrac{}{[x \leftrightarrow u] \vdash x : \overline{\llbracket \tau \rrbracket}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}}{x.\overline{\mathtt{some}}; [x \leftrightarrow u] \vdash x : \&\overline{\llbracket \tau \rrbracket}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}
$$

   b. **Rule** [$\mathtt{FS{:}var}^{!}$]**:** Then we have the following derivation provided $\eta_{ind} = \tau$:

$$
[\mathtt{FS{:}var}^{!}]\ \dfrac{[\mathtt{FS{:}var}^{\ell}]\ \dfrac{}{\Theta, x^! : \eta; x : \eta_{ind} \models x : \tau}}{\Theta, x^! : \eta; \text{-} \models x[ind] : \tau}
$$

By Def. 28, $\llbracket \Theta, x^! : \eta \rrbracket = \llbracket \Theta \rrbracket, x^! : \overline{\&_{\eta_i \in \eta}\{\mathtt{l}_i; \llbracket \eta_i \rrbracket\}}$, and by Fig. 8, $\llbracket x[ind] \rrbracket_u = \overline{x^!}?(x_i).x_i.l_{ind}; [x_i \leftrightarrow u]$. The thesis holds thanks to the following derivation:

$$
[\mathtt{Tcopy}]\ \dfrac{[\mathtt{T\oplus_i}]\ \dfrac{[(\mathtt{Tid})]\ \dfrac{}{[x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \overline{\llbracket \eta_{ind} \rrbracket}; x^! : \&_{\eta_i \in \eta}\{\mathtt{l}_i; \llbracket \eta_i \rrbracket\}, \llbracket \Theta \rrbracket}}{x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket, x_i : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}; x^! : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}, \llbracket \Theta \rrbracket}}{\overline{x^!}?(x_i).x_i.l_{ind}; [x_i \leftrightarrow u] \vdash u : \llbracket \tau \rrbracket; x^! : \oplus_{\eta_i \in \eta}\{\mathtt{l}_i; \overline{\llbracket \eta_i \rrbracket}\}, \llbracket \Theta \rrbracket}
$$

   c. **Rule** [$\mathtt{FS{:}weak}$]**:** Then we have the following derivation:

$$
[\mathtt{FS{:}weak}]\ \dfrac{\Theta; \Gamma \models M : \tau}{\Theta; \Gamma, x : \omega \models M[\leftarrow x] : \tau}
$$

By Def. 28, $\llbracket \Gamma, x : \omega \rrbracket = \llbracket \Gamma \rrbracket, x^{\ell} : \overline{\llbracket \omega \rrbracket_{(\sigma, i_1)}}$, and by Fig. 8,
$\llbracket M[\leftarrow x] \rrbracket_u = x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_i).(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; \llbracket M \rrbracket_u \mid x^{\ell}.\overline{\mathtt{none}})$.
By IH, we have $\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket$. The thesis holds thanks to the following derivation:

$$
[\mathtt{T\&^{x}_{d}}]\ \dfrac{[\mathtt{T\otimes}]\ \dfrac{[\mathtt{T\oplus^{x}_{w}}]\ \dfrac{[\mathtt{T\bot}]\ \dfrac{\llbracket M \rrbracket_u \vdash \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}{y_i.\mathtt{close}; \llbracket M \rrbracket_u \vdash y_i{:}\bot, \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}}{y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; \llbracket M \rrbracket_u \vdash y_i{:} \oplus \bot, \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket} \quad [\mathtt{T\&^{x}}]\ \dfrac{}{x^{\ell}.\overline{\mathtt{none}} \vdash x^{\ell} : \&A}}{\overline{x^{\ell}}(y_i).(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; \llbracket M \rrbracket_u \mid x^{\ell}.\overline{\mathtt{none}}) \vdash x^{\ell} : (\oplus\bot) \otimes (\&A), \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}}{\llbracket M[\leftarrow x] \rrbracket_u \vdash x^{\ell} : \&((\oplus\bot) \otimes (\&A)), \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}
$$

Since A is arbitrary, we can take $A = \mathbf{1}$ for $[\![\omega]\!]_{(\sigma,0)}$ and $A = \overline{((\&[\![\sigma]\!]) \parr ([\![\omega]\!]_{(\sigma,i-1)}))}$ for $[\![\omega]\!]_{(\sigma,i)}$ where $i > 0$, in both cases, the result follows.

d. Rule $[\mathtt{FS:abs\text{-}sh}]$**:**
Then $\mathbb{M} = \lambda x.(M[\widetilde{x} \leftarrow x])$, and the derivation is:

$$[\mathtt{FS{:}abs\text{-}sh}]\ \frac{\Theta, x^! : \eta; \Gamma, x : \sigma^k \models M[\widetilde{x} \leftarrow x] : \tau \quad x \notin \mathtt{dom}(\Gamma)}{\Theta; \Gamma \models \lambda x.(M[\widetilde{x} \leftarrow x]) : (\sigma^k, \eta) \to \tau}$$

By IH, we have $[\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash u : [\![\tau]\!], [\![\Gamma]\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!], x^! : \overline{[\![\eta]\!]}$, From Def. 26, it follows $[\![\lambda x.M[\widetilde{x} \leftarrow x]]\!]_u = u.\overline{\mathtt{some}}; u(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u$
We give the final derivation in parts. The first part we name Π_1 derived by:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash u : [\![\tau]\!], [\![\Gamma]\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!], x^! : \overline{[\![\eta]\!]}}{x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash x{:}\bot, u : [\![\tau]\!], [\![\Gamma]\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!], x^! : \overline{[\![\eta]\!]}}\ [\mathtt{T}\bot]}{x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash x{:}\bot, u : [\![\tau]\!], [\![\Gamma]\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}, x^! :?\overline{[\![\eta]\!]}; [\![\Theta]\!]}\ [\mathtt{T}?]}{x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash x : (?\overline{[\![\eta]\!]}) \parr (\bot), u : [\![\tau]\!], [\![\Gamma]\!], x^\ell : \overline{[\![\sigma^k]\!]_{(\sigma,i)}}; [\![\Theta]\!]}\ [\mathtt{T}\parr]}{x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \vdash x : \overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot)), u : [\![\tau]\!], [\![\Gamma]\!]; [\![\Theta]\!]}\ [\mathtt{T}\parr]$$

We take $P = x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u$ and continue the derivation:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \vdots\\ P \vdash x : \overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot)), u : [\![\tau]\!], [\![\Gamma]\!]; [\![\Theta]\!]\end{array}}{x.\overline{\mathtt{some}}; P \vdash x : \&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))), u : [\![\tau]\!], [\![\Gamma]\!]; [\![\Theta]\!]}\ [\mathtt{T}\&^x_d]}{u(x).x.\overline{\mathtt{some}}; P \vdash u : \&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!], [\![\Gamma]\!]; [\![\Theta]\!]}\ [\mathtt{T}\parr]}{u.\overline{\mathtt{some}}; u(x).x.\overline{\mathtt{some}}; P \vdash u : \&(\&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!]), [\![\Gamma]\!]; [\![\Theta]\!]}\ [\mathtt{T}\&^x_d]$$

By Definition 28 we have that $[\![(\sigma^k, \eta) \to \tau]\!] = \&(\&(\overline{[\![\sigma^k]\!]_{(\sigma,i)}} \parr ((?\overline{[\![\eta]\!]}) \parr (\bot))) \parr [\![\tau]\!])$.
Hence the case holds by $[\![\lambda x.M[\widetilde{x}^\ell \leftarrow x]]\!]_u \vdash u : [\![(\sigma^k, \eta) \to \tau]\!], [\![\Gamma]\!]; [\![\Theta]\!]$.

e. Rule $[\mathtt{FS:app}]$**:** Then $\mathbb{M} = M\ B$, where $B = C \star U$ and the derivation is:

$$[\mathtt{FS{:}app}]\ \frac{\Theta; \Gamma \models M : (\sigma^j, \eta) \to \tau \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma, \Delta \models M\ B : \tau}$$

By IH, we have both

- $[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![(\sigma^j, \eta) \to \tau]\!]; [\![\Theta]\!]$
- $[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![(\sigma^j, \epsilon) \to \tau]\!]; [\![\Theta]\!]$, by Lemma 63
- $[\![B]\!]_u \vdash [\![\Delta]\!], u : \overline{[\![(\sigma^k, \epsilon)]\!]_{(\tau_2,n)}}; [\![\Theta]\!]$, for some τ_2 and some n.

Therefore, from the fact that $\mathbb{M}$ is well-formed and Definitions 26 and 28, we have:

- $[\![M(C \star U)]\!]_u = \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x))$;
- $[\![(\sigma^j, \eta) \to \tau]\!] = \oplus(([\![\sigma^k]\!]_{(\tau_1,m)}) \otimes ((![\![\eta]\!]))$, for some τ_1 and some m.

Also, since $[\![B]\!]_u \vdash [\![\Delta]\!], u : [\![(\sigma^k, \epsilon)]\!]_{(\tau_2,n)}$, we have the following derivation Π_i:

$$\dfrac{[\texttt{T}\otimes]\ \dfrac{[\![C_i \star U]\!]_x \vdash [\![\Delta]\!], x : [\![(\sigma^k,\epsilon)]\!]_{(\tau_2,n)}; [\![\Theta]\!] \qquad [\texttt{Tid}]\ \dfrac{}{[v \leftrightarrow u] \vdash v : \overline{[\![\tau]\!]}, u : [\![\tau]\!]}}{\overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x) \vdash [\![\Delta]\!], v : [\![(\sigma^k,\epsilon)]\!]_{(\tau_2,n)} \otimes \overline{[\![\tau]\!]}, u : [\![\tau]\!]; [\![\Theta]\!]}}{v.\texttt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x) \vdash [\![\Delta]\!], v : \oplus([\![(\sigma^k,\epsilon)]\!]_{(\tau_2,n)} \otimes \overline{[\![\tau]\!]}), u : [\![\tau]\!]; [\![\Theta]\!]}\ [\texttt{T}\oplus^{\texttt{v}}_{\texttt{w}}]$$

Notice that $\oplus([\![(\sigma^k,\epsilon)]\!]_{(\tau_2,n)} \otimes \overline{[\![\tau]\!]}) = \overline{[\![(\sigma^k,\epsilon) \to \tau]\!]}$. Therefore, by one application of [Tcut] we obtain the derivations ∇_i, for each $C_i \in \mathsf{PER}(C)$:

$$[\texttt{Tcut}]\ \dfrac{[\![M]\!]_v \vdash [\![\Gamma]\!], v : [\![(\sigma^j,\epsilon) \to \tau]\!]; [\![\Theta]\!] \qquad \Pi_i}{(\nu v)([\![M]\!]_v \mid v.\texttt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![B_i]\!]_x)) \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]; [\![\Theta]\!]}$$

In order to apply [Tcut], we must have that $[\![\sigma^j]\!]_{(\tau_1,m)} = [\![\sigma^k]\!]_{(\tau_2,n)}$, therefore, the choice of τ_1, τ_2, n and m, will consider the different possibilities for j and k, as in Proposition 62. We can then conclude that $[\![MB]\!]_u \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]; [\![\Theta]\!]$:

$$[\texttt{T}\&]\ \dfrac{\text{For each } C_i \in \mathsf{PER}(C) \qquad \nabla_i}{\bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\texttt{some}_{u,\mathsf{fv}(B)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![B_i]\!]_x)) \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]; [\![\Theta]\!]}$$

and the result follows.

f. Rule [FS : share]**:** Then $\mathbb{M} = M[x_1, \ldots x_k \leftarrow x]$ and the derivation is:

$$[\texttt{FS : share}]\ \dfrac{\Theta; \Delta, x_1 : \sigma, \cdots, x_k : \sigma \models M : \tau \quad x \notin \Delta \quad k \neq 0}{\Theta; \Delta, x : \sigma_k \models M[x_1, \cdots, x_k \leftarrow x] : \tau}$$

To simplify the proof we will consider $k = 1$ (the case in which $k > 1$ follows similarly). By IH, we have $[\![M]\!]_u \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!]; [\![\Theta]\!]$. From Definitions 26 and 28, it follows

- $[\![\Delta, x_1 : \sigma]\!] = [\![\Delta]\!], x_1^\ell : \&\overline{[\![\sigma]\!]}$.
- $[\![M[x_1, \leftarrow x]]\!]_u = x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_1).(y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \mid x^\ell.\overline{\texttt{some}}; x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)};$ $x^\ell(x_1).x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_2).(y_2.\texttt{some}_{u,\mathsf{lfv}(M)}; y_2.\texttt{close}; [\![M]\!]_u \mid x^\ell.\overline{\texttt{none}}))$

We shall split the expression into two parts:

$$N_1 = x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_2).(y_2.\texttt{some}_{u,\mathsf{lfv}(M)}; y_2.\texttt{close}; [\![M]\!]_u \mid x^\ell.\overline{\texttt{none}})$$
$$N_2 = x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_1).(y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \mid x^\ell.\overline{\texttt{some}}; x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)}; x^\ell(x_1).N_1)$$

and we obtain the derivation for term N_1 as follows where we omit $; [\![\Theta]\!]$:

$$[\texttt{T}\&^{\texttt{x}}_{\texttt{d}}]\ \dfrac{[\texttt{T}\otimes]\ \dfrac{[\texttt{T}\oplus^{\texttt{x}}_{\texttt{w}}]\ \dfrac{[\texttt{T}\bot]\ \dfrac{[\![M]\!]_u \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!]}{y_2.\texttt{close}; [\![M]\!]_u \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!], y_2{:}\bot}}{y_2.\texttt{some}_{u,\mathsf{lfv}(M)}; y_2.\texttt{close}; [\![M]\!]_u \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!], y_2{:} \oplus \bot} \qquad [\texttt{T}\&^{\texttt{x}}]\ \dfrac{}{x^\ell.\overline{\texttt{none}} \vdash x^\ell : \& A}}{\overline{x^\ell}(y_2).(y_2.\texttt{some}_{u,\mathsf{lfv}(M)}; y_2.\texttt{close}; [\![M]\!]_u \mid x^\ell.\overline{\texttt{none}}) \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!], x^\ell : (\oplus\bot) \otimes (\& A)}}{\underbrace{x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_2).(y_2.\texttt{some}_{u,\mathsf{lfv}(M)}; y_2.\texttt{close}; [\![M]\!]_u \mid x^\ell.\overline{\texttt{none}})}_{N_1} \vdash [\![\Delta, x_1 : \sigma]\!], u : [\![\tau]\!], x^\ell : \overline{[\![\omega]\!]_{(\sigma,i)}}}$$

Notice that the last rule applied $[\texttt{T}\&^{\texttt{x}}_{\texttt{d}}]$ assigns $x : \overline{\&((\oplus\bot) \otimes (\& A))}$. Again, since A is arbitrary, we can take $A = \oplus((\&\overline{[\![\sigma]\!]}) \mathbin{⅋} ([\![\omega]\!]_{(\sigma,i-1)}))$, obtaining $x : \overline{[\![\omega]\!]_{(\sigma,i)}}$.

In order to obtain a type derivation for N_2, consider the derivation Π_1:

$$[\texttt{T}\parr]\ \frac{N_1 \vdash \llbracket \Delta \rrbracket, x_1 : \&\overline{\llbracket \sigma \rrbracket}, u : \llbracket \tau \rrbracket, x^\ell : \overline{\llbracket \omega \rrbracket_{(\sigma,i)}}}{x^\ell(x_1).N_1 \vdash \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket, x^\ell : (\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}})}$$

$$[\texttt{T}\oplus^{\texttt{x}}_{\texttt{w}}]\ \frac{x^\ell(x_1).N_1 \vdash \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket, x^\ell : (\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}})}{x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)}; x^\ell(x_1).N_1 \vdash \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket, x^\ell : \oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}}))}$$

$$[\texttt{T}\&^{\texttt{x}}_{\texttt{d}}]\ \frac{x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)}; x^\ell(x_1).N_1 \vdash \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket, x^\ell : \oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}}))}{x^\ell.\overline{\texttt{some}}; x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)}; x^\ell(x_1).N_1 \vdash \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket, x^\ell : \&\oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}}))}$$

We take $P_1 = x^\ell.\overline{\texttt{some}}; x^\ell.\texttt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)}; x^\ell(x_1).N_1$ and $\Gamma_1 = \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket$ and continue the derivation of N_2

$$[\texttt{T}\cdot]\ \frac{}{0 \vdash \texttt{-}; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\bot]\ \frac{0 \vdash \texttt{-}; \llbracket \Theta \rrbracket}{y_1.\texttt{close}; 0 \vdash y_1 : \bot; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\oplus^{\texttt{x}}_{\texttt{w}}]\ \frac{y_1.\texttt{close}; 0 \vdash y_1 : \bot; \llbracket \Theta \rrbracket}{y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \vdash y_1 : \oplus\bot; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\otimes]\ \frac{y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \vdash y_1 : \oplus\bot; \llbracket \Theta \rrbracket \qquad \begin{array}{c}\Pi_1\\ \vdots\\ P_1 \vdash \Gamma_1, x^\ell : \&\oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}})); \llbracket \Theta \rrbracket\end{array}}{\overline{x^\ell}(y_1).(y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \mid P_1) \vdash \Gamma_1, x^\ell : (\oplus\bot) \otimes (\&\oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}}))); \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\&^{\texttt{x}}_{\texttt{d}}]\ \frac{\overline{x^\ell}(y_1).(y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \mid P_1) \vdash \Gamma_1, x^\ell : (\oplus\bot) \otimes (\&\oplus((\&\overline{\llbracket \sigma \rrbracket}) \parr (\overline{\llbracket \omega \rrbracket_{(\sigma,i)}}))); \llbracket \Theta \rrbracket}{\underbrace{x^\ell.\overline{\texttt{some}}.\overline{x^\ell}(y_1).(y_1.\texttt{some}_\emptyset; y_1.\texttt{close}; 0 \mid P_1)}_{N_2} \vdash \Gamma_1, x^\ell : \overline{\llbracket \sigma \wedge \omega \rrbracket_{(\sigma,i)}}; \llbracket \Theta \rrbracket}$$

Hence the theorem holds for this case.

g. Rule [$\texttt{FS}:\texttt{ex-sub}$]**:** Then $\mathbb{M} = (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle$ and

$$[\texttt{FS:ex-sub}]\ \frac{\Theta, x^! : \eta; \Gamma, x : \sigma^j \models M[\widetilde{x} \leftarrow x] : \tau \qquad \Theta; \Delta \models B : (\sigma^k, \epsilon) \qquad \eta \propto \epsilon}{\Theta; \Gamma, \Delta \models (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x \rangle\!\rangle : \tau}$$

By Proposition 62 and IH we have:

$$\begin{array}{rl}
\llbracket M[x_1, \cdots, x_k \leftarrow x] \rrbracket_u & \vdash \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket, x^! : \overline{\llbracket \eta \rrbracket} \\
\llbracket M[x_1, \cdots, x_k \leftarrow x] \rrbracket_u & \vdash \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket, x^! : \overline{\llbracket \epsilon \rrbracket}, \text{ by Lemma 63} \\
\llbracket B \rrbracket_x & \vdash \llbracket \Delta \rrbracket, x : \llbracket (\sigma^k, \epsilon) \rrbracket_{(\tau,m)}; \llbracket \Theta \rrbracket
\end{array}$$

From Def. 26, we have

$$\llbracket M[\widetilde{x} \leftarrow x]\ \langle\!\langle B/x \rangle\!\rangle \rrbracket_u = \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\texttt{some}}; x(x^\ell).x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \mid \llbracket C_i \star U \rrbracket_x)$$

Therefore, for each $B_i \in \mathsf{PER}(B)$, we obtain the following derivation Π_i:

$$[\texttt{T}\bot]\ \frac{\llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket, x^! : \overline{\llbracket \epsilon \rrbracket}}{x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x{:}\bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket, x^! : \overline{\llbracket \epsilon \rrbracket}}$$

$$[\texttt{T}?]\ \frac{x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x{:}\bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket, x^! : \overline{\llbracket \epsilon \rrbracket}}{x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x{:}\bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket, x^! : ?\overline{\llbracket \epsilon \rrbracket}; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\parr]\ \frac{x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x{:}\bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket, x^! : ?\overline{\llbracket \epsilon \rrbracket}; \llbracket \Theta \rrbracket}{x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x : (?\overline{\llbracket \epsilon \rrbracket}) \parr \bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\parr]\ \frac{x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x : (?\overline{\llbracket \epsilon \rrbracket}) \parr \bot, \llbracket \Gamma \rrbracket, x^\ell : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}}, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}{x(x^\ell).x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}} \parr ((?\overline{\llbracket \epsilon \rrbracket}) \parr \bot), \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}$$

$$[\texttt{T}\&^{\texttt{x}}_{\texttt{d}}]\ \frac{x(x^\ell).x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x : \overline{\llbracket \sigma^j \rrbracket_{(\tau,n)}} \parr ((?\overline{\llbracket \epsilon \rrbracket}) \parr \bot), \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}{x.\overline{\texttt{some}}; x(x^\ell).x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u \vdash x : \overline{\llbracket (\sigma^j, \epsilon) \rrbracket_{(\tau,n)}}, \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}$$

We take $P_1 = x.\overline{\texttt{some}}; x(x^\ell).x(x^!).x.\texttt{close}; \llbracket M[\widetilde{x} \leftarrow x] \rrbracket_u$ and continue the derivation of Π_i

$$[\texttt{Tcut}]\ \frac{P_1 \vdash x : \overline{\llbracket (\sigma^j, \epsilon) \rrbracket_{(\tau,n)}}, \llbracket \Gamma \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket \qquad \llbracket C_i \star U \rrbracket_x \vdash \llbracket \Delta \rrbracket, x : \llbracket (\sigma^k, \epsilon) \rrbracket_{(\tau,m)}; \llbracket \Theta \rrbracket}{(\nu x)(P_1 \mid \llbracket C_i \star U \rrbracket_x) \vdash \llbracket \Gamma \rrbracket, \llbracket \Delta \rrbracket, u : \llbracket \tau \rrbracket; \llbracket \Theta \rrbracket}$$

We must have that $[\![\sigma^j]\!]_{(\tau,m)} = [\![\sigma^k]\!]_{(\tau,n)}$ which by our restrictions allows. Therefore, from Π_i and multiple applications of [T&] it follows that

$$[\texttt{T\&}]\ \frac{\forall \bigoplus_{C_i \in \mathsf{PER}(C)} \qquad \Pi_i}{\bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(P_1 \mid [\![C_i \star U]\!]_x) \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]; [\![\Theta]\!]}$$

that is, $[\![M[x_1 \leftarrow x]\ \langle\!\langle B/x \rangle\!\rangle]\!] \vdash [\![\Gamma, \Delta]\!], u : [\![\tau]\!]; [\![\Theta]\!]$ and the result follows.

h. Rule [FS:ex-sub$^\ell$]**:** Then $\mathbb{M} = M\langle\!| N/x |\!\rangle$ and

$$[\texttt{FS:ex-sub}^\ell]\ \frac{\Theta; \Gamma, x : \sigma \models M : \tau \quad \Theta; \Delta \models N : \sigma}{\Theta; \Gamma, \Delta \models M\langle\!| N/x |\!\rangle : \tau}$$

By IH we have both

$$[\![N]\!]_x \vdash [\![\Delta]\!], x : [\![\sigma]\!]; [\![\Theta]\!]$$

$$[\![M]\!]_u \vdash [\![\Gamma]\!], x : \&\overline{[\![\sigma]\!]}, u : [\![\tau]\!]; [\![\Theta]\!]$$

From Definition 26, $[\![M\langle\!| N/x |\!\rangle]\!]_u = (\nu x)([\![M]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x)$ and

$$[\texttt{TCut}]\ \frac{[\![M]\!]_u \vdash [\![\Gamma]\!], x : \&\overline{[\![\sigma]\!]}, u : [\![\tau]\!]; [\![\Theta]\!] \qquad [\texttt{T}\oplus^{\mathsf{x}}]\ \dfrac{[\![N]\!]_x \vdash [\![\Delta]\!], x : [\![\sigma]\!]; [\![\Theta]\!]}{x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x \vdash [\![\Delta]\!], x :: \oplus[\![\sigma]\!]}}{(\nu x)([\![M]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x) \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]}$$

Observe that for the application of rule [TCut] we used the fact that $\overline{\oplus[\![\sigma]\!]} = \&\overline{[\![\sigma]\!]}$. Therefore, $[\![M\langle\!| N/x |\!\rangle]\!]_u \vdash [\![\Gamma]\!], [\![\Delta]\!], u : [\![\tau]\!]$ and the result follows.

i. Rule [FS:ex-sub$^!$]**:** Then $\mathbb{M} = M\|U/x\|$ and

$$[\texttt{FS:ex-sub}^!]\ \frac{\Theta, x^! : \eta; \Gamma \models M : \tau \quad \Theta; \text{-} \models U : \eta}{\Theta; \Gamma \models M\|U/x\| : \tau}$$

By IH we have both

$$\begin{aligned} [\![U]\!]_{x_i} &\vdash x_i : [\![\eta]\!]; [\![\Theta]\!] \\ [\![M]\!]_u &\vdash [\![\Gamma]\!], u : [\![\tau]\!]; x^! : \overline{[\![\eta]\!]}, [\![\Theta]\!] \end{aligned}$$

From Definition 26, $[\![M\|U/x\|]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i})$ and

$$[\texttt{TCut}]\ \frac{[\texttt{T?}]\ \dfrac{[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![\tau]\!]; x^! : \overline{[\![\eta]\!]}, [\![\Theta]\!]}{[\![M]\!]_u \vdash [\![\Gamma]\!], u : [\![\tau]\!], x^! : ?\overline{[\![\eta]\!]}; [\![\Theta]\!]} \quad [\texttt{T!}]\ \dfrac{[\![U]\!]_{x_i} \vdash x_i : [\![\eta]\!]; [\![\Theta]\!]}{!x^!.(x_i).[\![U]\!]_{x_i} \vdash x^! : ![\![\eta]\!]; [\![\Theta]\!]}}{(\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i}) \vdash [\![\Gamma]\!], u : [\![\tau]\!]; [\![\Theta]\!]}$$

Observe that for the application of rule [TCut] we used the fact that $\overline{![\![\eta]\!]} = ?\overline{[\![\eta]\!]}$. Therefore, $[\![M\|U/x\|]\!]_u \vdash [\![\Gamma]\!], u : [\![\tau]\!]; [\![\Theta]\!]$ and the result follows.

j. Rule [FS : fail]**:** Then $\mathbb{M} = \mathtt{fail}^{\widetilde{x}}$ where $\widetilde{x} = x_1, \cdots, x_n$ and

$$[\texttt{FS:fail}]\ \frac{\mathtt{dom}(\Gamma) = \widetilde{x}}{\Theta; \Gamma \models \mathtt{fail}^{\widetilde{x}} : \tau}$$

From Definition 26, $[\![\mathtt{fail}^{x_1,\cdots,x_n}]\!]_u = u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid \cdots \mid x_k.\overline{\mathtt{none}}$ and

$$\dfrac{[\mathtt{T}\&^{\mathtt{u}}]\ \dfrac{}{u.\overline{\mathtt{none}} \vdash u : [\![\tau]\!]; [\![\Theta]\!]} \quad \dfrac{[\mathtt{T}\&^{\mathtt{x_1}}]\ \dfrac{}{x_1.\overline{\mathtt{none}} \vdash_1 : \&\overline{[\![\sigma_1]\!]}; [\![\Theta]\!]} \quad \dfrac{[\mathtt{T}\&^{\mathtt{x_n}}]\ \dfrac{}{x_n.\overline{\mathtt{none}} \vdash x_n : \&\overline{[\![\sigma_n]\!]}; [\![\Theta]\!]}}{\vdots}}{x_1.\overline{\mathtt{none}} \mid \cdots \mid x_k.\overline{\mathtt{none}} \vdash x_1 : \&\overline{[\![\sigma_1]\!]}, \cdots, x_n : \&\overline{[\![\sigma_n]\!]}; [\![\Theta]\!]}}{u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid \cdots \mid x_k.\overline{\mathtt{none}} \vdash x_1 : \&\overline{[\![\sigma_1]\!]}, \cdots, x_n : \&\overline{[\![\sigma_n]\!]}, u : [\![\tau]\!]; [\![\Theta]\!]}\ [\mathtt{T}\mid]$$

Thus, $[\![\mathtt{fail}^{x_1,\cdots,x_n}]\!]_u \vdash x_1 : \&\overline{[\![\sigma_1]\!]}, \cdots, x_n : \&\overline{[\![\sigma_n]\!]}, u : [\![\tau]\!]; [\![\Theta]\!]$ and the result follows.

k. Rule $[\mathtt{FS}:\mathtt{sum}]$: This case follows easily by IH.

◀

F.2 Operational Correspondence: Completeness and Soundness

▶ **Proposition 65.** *Let N be a well-formed linearly closed $u\widehat{\lambda}^{\lightning}_{\oplus}$-term with $\mathsf{head}(N) = x$ (x denoting either linear or unrestricted occurrence of x) such that $\mathsf{lfv}(N) = \emptyset$ and N does not fail, that is, there is no $Q \in u\widehat{\lambda}^{\lightning}_{\oplus}$ for which there is a reduction $N \longrightarrow_{[\mathtt{RS:Fail}]} Q$. Then,*

$$[\![N]\!]_u \longrightarrow^* \bigoplus_{i\in I} (\nu \widetilde{y})([\![x]\!]_n \mid P_i)$$

for some index set I, names $\widetilde{y}$ and n, and processes P_i.

Proof. By induction on the structure of N.

1. $N = x$ or $N = x[j]$:

   These cases are trivial, and follow taking $I = \emptyset$ and $\widetilde{y} = \emptyset$.

2. $N = (M\ B)$:

   Then $\mathsf{head}(M\ B) = \mathsf{head}(M) = x$ then

$$[\![N]\!]_u = [\![M\ B]\!]_u = \bigoplus_{B_i \in \mathsf{PER}(B)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(B)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![B_i^x]\!]))$$

   and the proof follows by induction on $[\![M]\!]_u$.

3. $N = (M[\widetilde{y} \leftarrow y])\langle\!\langle C \star U / y \rangle\!\rangle$:

   Then $\mathsf{head}((M[\widetilde{y} \leftarrow y])\langle\!\langle C \star U / y \rangle\!\rangle) = \mathsf{head}((M[\widetilde{y} \leftarrow y])) = x$. As $N \longrightarrow_{[\mathtt{R}]}$ where $[\mathtt{R}] \neq [\mathtt{RS}:\mathtt{Fail}]$ we must have that $\mathsf{size}(\widetilde{y}) = \mathsf{size}(C)$. Thus,

$$
\begin{aligned}
[\![N]\!]_u &= [\![(M[\widetilde{y} \leftarrow y])\langle\!\langle C \star U / y\rangle\!\rangle]\!]_u \\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu y)(y.\overline{\mathtt{some}}; y(y^\ell).y(y^!).y.\mathtt{close}; [\![M[\widetilde{y} \leftarrow y]]\!]_u \mid [\![C_i \star U]\!]_y) \\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu y)(y.\overline{\mathtt{some}}; y(y^\ell).y(y^!).y.\mathtt{close}; [\![M[\widetilde{y} \leftarrow y]]\!]_u \mid \\
&\qquad y.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{y}(y^\ell).([\![C_i]\!]_{y^\ell} \mid \overline{y}(y^!).(!y^!.(y_i).[\![U]\!]_{y_i} \mid y.\overline{\mathtt{close}}))) \\
&\longrightarrow^* \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu y^\ell, y^!)([\![M[\widetilde{y} \leftarrow y^\ell]]\!]_u \mid [\![C_i]\!]_{y^\ell} \mid !y^!.(y_i).[\![U]\!]_{y_i}) \\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu y^\ell, y^!)(y^\ell.\overline{\mathtt{some}}.\overline{y^\ell}(z_1).(z_1.\mathtt{some}_{\emptyset}; z_1.\mathtt{close}; 0 \mid y^\ell.\overline{\mathtt{some}}; \\
&\quad y^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus y_1,\cdots,y_n)}; y^\ell(y_1).\cdots y^\ell.\overline{\mathtt{some}}.\overline{y^\ell}(z_n).(z_n.\mathtt{some}_{\emptyset}; z_n.\mathtt{close}; 0 \\
&\quad \mid y^\ell.\overline{\mathtt{some}}; y^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus y_n)}; y^\ell(y_n).y^\ell.\overline{\mathtt{some}}; \overline{y^\ell}(z_{n+1}).(z_{n+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; \\
&\quad z_{n+1}.\mathtt{close}; [\![M]\!]_u \mid y^\ell.\overline{\mathtt{none}})) \cdots) \mid y^\ell.\mathtt{some}_{\mathsf{lfv}(C)}; y^\ell(z_1).y^\ell.\mathtt{some}_{z_1,\mathsf{lfv}(C)}; y^\ell.\overline{\mathtt{some}}; \\
&\quad \overline{y^\ell}(y_1).(y_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{y_1} \mid \cdots y^\ell.\mathtt{some}_{\mathsf{lfv}(C_i(n))}; y^\ell(z_n).y^\ell.\mathtt{some}_{z_n,\mathsf{lfv}(C_i(n))}; \\
&\quad y^\ell.\overline{\mathtt{some}}; \overline{y^\ell}(y_n).(y_n.\mathtt{some}_{\mathsf{lfv}(C_i(n))}; [\![C_i(n)]\!]_{y_n} \mid y^\ell.\mathtt{some}_{\emptyset}; y^\ell(z_{n+1}).(z_{n+1}.\overline{\mathtt{some}}; \\
&\quad z_{n+1}.\overline{\mathtt{close}} \mid y^\ell.\mathtt{some}_{\emptyset}; y^\ell.\overline{\mathtt{none}}) \mid z_1.\overline{\mathtt{none}}) \mid \cdots \mid z_n.\overline{\mathtt{none}}) \mid !y^!.(y_i).[\![U]\!]_{y_i}) \\
&\longrightarrow^* \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu \widetilde{y}, y^!)([\![M]\!]_u \mid y_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{y_1} \mid \cdots \mid y_n.\mathtt{some}_{\mathsf{lfv}(C_i(n))}; \\
&\quad [\![C_i(n)]\!]_{y_n} \mid !y^!.(y_i).[\![U]\!]_{y_i})
\end{aligned}
$$

and the result follows by induction on $[\![M]\!]_u$.

4. $N = M\langle\!| N'/y |\!\rangle$ and $N = M\lfloor\!\lfloor u/y \rceil\!\rceil$:
   These cases follow easily by induction on $[\![M]\!]_u$.

◀

F.2.1 Completeness

Here again, because of the diamond property (Proposition 33), it suffices to consider a completeness result based on a single reduction step in $u\widehat{\lambda}^{\lightning}_{\oplus}$:

▶ **Notation 8.** *We use the notation* $\mathsf{lfv}(M).\overline{\mathtt{none}}$ *and* $\widetilde{x}.\overline{\mathtt{none}}$ *where* $\mathsf{lfv}(M)$ *or* $\widetilde{x}$ *are equal to* $x_1, \cdots, x_k$ *to describe a process of the form* $x_1.\overline{\mathtt{none}} \mid \cdots \mid x_k.\overline{\mathtt{none}}$

▶ **Theorem 66** (Well Formed Operational Completeness)**.** *Let* $\mathbb{N}$ *and* $\mathbb{M}$ *be well-formed, linearly closed* $u\widehat{\lambda}^{\lightning}_{\oplus}$ *expressions. If* $\mathbb{N} \longrightarrow \mathbb{M}$ *then there exists* Q *such that* $[\![\mathbb{N}]\!]_u \longrightarrow^* Q \equiv [\![\mathbb{M}]\!]_u$.

Proof. By induction on the reduction rule applied to infer $\mathbb{N} \longrightarrow \mathbb{M}$. We have ten cases.

1. **Case** $[\mathtt{RS} : \mathtt{Beta}]$:
   Then $\mathbb{N} = (\lambda x.(M[\widetilde{x} \leftarrow x]))B \longrightarrow (M[\widetilde{x} \leftarrow x])\langle\!\langle B/x\rangle\!\rangle = \mathbb{M}$, where $B = C \star U$. Notice

that

$$
\begin{aligned}
[\![\mathbb{N}]\!]_u &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![\lambda x.(M[\widetilde{x} \leftarrow x])]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x))\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(v.\overline{\mathtt{some}}; v(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_v\\
&\quad \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([\![C_i \star U]\!]_x \mid [v \leftrightarrow u]))\\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(v(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_v \mid \overline{v}(x).([\![C_i \star U]\!]_x\\
&\qquad \mid [v \leftrightarrow u]))\\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v, x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_v \mid [\![C_i \star U]\!]_x \mid [v \leftrightarrow u])\\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_v \mid [\![C_i \star U]\!]_x) = [\![\mathbb{M}]\!]_u
\end{aligned}
$$

and the result follows.

2. **Case** [$\mathtt{RS}$: $\mathtt{Ex\text{-}Sub}$]:
Then $\mathbb{N} = M[x_1,\cdots,x_k \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle$, with $C = \wr M_1 \int \cdots \wr M_k \int$, $k \geq 0$ and $M \neq \mathtt{fail}^{\widetilde{y}}$. The reduction is

$$
\mathbb{N} = M[x_1,\cdots,x_k \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle \longrightarrow \sum_{C_i \in \mathsf{PER}(C)} M \langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(k)/x_k |\!\rangle \lfloor U/x \rceil = \mathbb{M}.
$$

We detail the encodings of $[\![\mathbb{N}]\!]_u$ and $[\![\mathbb{M}]\!]_u$. To simplify the proof, we will consider $k = 1$ (the case in which $k > 1$ is follows analogously, similarly the case of $k = 0$ is contained within the proof of $k = 1$).
On the one hand, we have:

$$
\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![M[x_1 \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle]\!]_u\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[x_1 \leftarrow x]]\!]_u \mid [\![C_i \star U]\!]_x)\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[x_1 \leftarrow x]]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^\ell).\\
&\qquad ([\![C_i]\!]_{x^\ell} \mid \overline{x}(x^!).(!x^!.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}}))) \qquad (:= P_{\mathbb{N}})
\end{aligned}
$$

Note that

$$
\begin{aligned}
P_{\mathbb{N}} &\longrightarrow^* \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)([\![M[x_1 \leftarrow x]]\!]_u \mid [\![C_i]\!]_{x^\ell} \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)(x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_1).(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_1)};\\
&\quad x^\ell(x_1).x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(y_2).(y_2.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_2.\mathtt{close}; [\![M]\!]_u \mid x^\ell.\overline{\mathtt{none}})) \mid x^\ell.\mathtt{some}_{\mathsf{lfv}(B_i(1))};\\
&\quad x^\ell(y_1).x^\ell.\mathtt{some}_{y_1,\mathsf{lfv}(C_i(1))}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_1).(x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid x^\ell.\\
&\quad \mathtt{some}_\emptyset; x^\ell(y_2).(y_2.\overline{\mathtt{some}}; y_2.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}})) \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow^* \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x_1, x^!)([\![M]\!]_u \mid x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid !x^!.(x_i).[\![U]\!]_{x_i}) = [\![\mathbb{M}]\!]_u
\end{aligned}
$$

and the result follows.

3. **Case** [$\texttt{RS:Fetch}^{\ell}$]**:**

Then we have $\mathbb{N} = M\langle\!|N/x|\!\rangle$ with $\mathsf{head}(M) = x$ and $\mathbb{N} \longrightarrow M\{\!|N/x|\!\} = \mathbb{M}$. Note that

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![M\langle\!|N/x|\!\rangle]\!]_u \\
&= (\nu x)([\![M]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)};[\![N]\!]_x) \\
&\longrightarrow^* (\nu x)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![x]\!]_j \mid P_i) \mid x.\mathtt{some}_{\mathsf{lfv}(N)};[\![N]\!]_x) \qquad (*) \\
&= (\nu x)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![x]\!]_j \mid P_i) \mid x.\mathtt{some};[\![N]\!]_x) \\
&\longrightarrow (\nu x)(\bigoplus_{i\in I}(\nu \widetilde{y})([x \leftrightarrow j] \mid P_i) \mid [\![N]\!]_x) \\
&\longrightarrow \bigoplus_{i\in I}(\nu \widetilde{y})(P_i \mid [\![N]\!]_j) = [\![\mathbb{M}]\!]_u
\end{aligned}$$

where the reductions denoted by $(*)$ are inferred via Proposition 65, and the result follows.

4. **Case** [$\texttt{RS:Fetch}^{!}$]**:**

Then, $\mathbb{N} = M\lfloor\!\lfloor U/x\rceil\!\rceil$ with $\mathsf{head}(M) = x^![k]$, $U_i = \wr N \int^!$ and $\mathbb{N} \longrightarrow M\{\!|N/x^!|\!\}\lfloor\!\lfloor U/x\rceil\!\rceil = \mathbb{M}$. Note that

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![M\lfloor\!\lfloor U/x\rceil\!\rceil]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&\longrightarrow^* (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![x^![k]]\!]_j \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \qquad (*) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})(\overline{x^!}?(x_k).x_k.l_i;[x_k \leftrightarrow j] \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \qquad (*) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})((\nu x_k)(x_k.l_i;[x_k \leftrightarrow j] \mid [\![U]\!]_{x_k}) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})((\nu x_k)(x_k.l_i;[x_k \leftrightarrow j] \mid x_k.case(i.[\![U_i]\!]_x)) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![\wr N \int^!]\!]_j) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![N]\!]_j) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) = [\![\mathbb{M}]\!]_u
\end{aligned}$$

where the reductions denoted by $(*)$ are inferred via Proposition 65.

5. **Cases** [$\texttt{RS : TCont}$] **and** [$\texttt{RS : ECont}$]**:**

These cases follow by IH.

6. **Case** [$\texttt{RS:Fail}^{\ell}$]**:**

Then, $\mathbb{N} = M[x_1,\cdots,x_k \leftarrow x]\ \langle\!\langle C \star U/x\rangle\!\rangle$ with $k \neq \mathsf{size}(C)$ and

$\mathbb{N} \longrightarrow \sum_{C_i\in\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{y}} = \mathbb{M}$, where $\widetilde{y} = (\mathsf{lfv}(M) \setminus \{x_1,\cdots,x_k\}) \cup \mathsf{lfv}(C)$.

Let $\mathsf{size}(C) = l$ and we assume that $k > l$ (proceed similarly for $k > l$). Hence $k = l + m$ for some $m \geq 1$, and

$$
\begin{aligned}
[\![N]\!]_u =& [\![M[x_1,\cdots,x_k \leftarrow x]\ \langle\!\langle C \star U/x \rangle\!\rangle]\!]_u \\
=& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \mid \\
& x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^\ell).([\![C]\!]_{x^\ell} \mid \overline{x}(x^!).(!x^!.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}}))) \\
\longrightarrow^*& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)([\![M[\widetilde{x} \leftarrow x]]\!]_u \mid [\![C]\!]_{x^\ell} \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
=& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)(x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_1).(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; \\
& x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus\widetilde{x})}; x^\ell(x_1).\cdots x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_k).(y_k.\mathtt{some}_\emptyset; y_k.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; \\
& x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_k)}; x^\ell(x_k).x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(y_{k+1}).(y_{k+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{k+1}.\mathtt{close}; [\![M]\!]_u \\
& \mid x^\ell.\overline{\mathtt{none}})) \cdots) \mid x^\ell.\mathtt{some}_{\mathsf{lfv}(C)}; x^\ell(y_1).x^\ell.\mathtt{some}_{y_1,\mathsf{lfv}(C)}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_1).(x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; \\
& [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid \cdots x^\ell.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; x^\ell(y_l).x^\ell.\mathtt{some}_{y_l,\mathsf{lfv}(C_i(l))}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_l). \\
& (x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid y_l.\overline{\mathtt{none}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \\
& \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}))) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \qquad (:= P_{\mathbb{N}}) \\
\longrightarrow^*& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!, y_1, x_1, \cdots y_l, x_l)(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid \cdots \mid y_l.\mathtt{some}_\emptyset; y_l.\mathtt{close}; 0 \\
& x.\overline{\mathtt{some}}.\overline{x^\ell}(y_{l+1}).(y_{l+1}.\mathtt{some}_\emptyset; y_{l+1}.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_{l+1},\cdots,x_k)}; \\
& x^\ell(x_{l+1}).\cdots x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_k).(y_k.\mathtt{some}_\emptyset; y_k.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_k)}; \\
& x^\ell(x_k).x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(y_{k+1}).(y_{k+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{k+1}.\mathtt{close}; [\![M]\!]_u \mid x^\ell.\overline{\mathtt{none}})) \cdots) \mid \\
& x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid y_1.\overline{\mathtt{none}} \mid \cdots \mid y_l.\overline{\mathtt{none}} \\
& x^\ell.\mathtt{some}_\emptyset; x^\ell(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
\longrightarrow& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^!, x_1, \cdots x_l)(u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid \cdots \mid x_l.\overline{\mathtt{none}} \mid (\mathsf{lfv}(M) \setminus x_1, \cdots, x_k). \\
& \overline{\mathtt{none}} \mid x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
\longrightarrow^*& \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^!)(u.\overline{\mathtt{none}} \mid (\mathsf{lfv}(M) \setminus x_1, \cdots, x_k).\overline{\mathtt{none}} \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
\equiv& \bigoplus_{C_i \in \mathsf{PER}(C)} u.\overline{\mathtt{none}} \mid (\mathsf{lfv}(M) \setminus x_1, \cdots, x_k).\overline{\mathtt{none}} = [\![\mathbb{M}]\!]_u
\end{aligned}
$$

and the result follows.

7. **Case** [$\mathtt{RS{:}Fail}^!$]**:**

Then, $\mathbb{N} = M\llbracket U/x \rrbracket$ with $\mathsf{head}(M) = x[i]$, $U_i = \mathtt{1}^!$ and $\mathbb{N} \longrightarrow M\{\!|\mathtt{fail}^\emptyset/x^!|\!\}\llbracket U/x \rrbracket$, where $\widetilde{y} = \mathsf{lfv}(M)$. Notice that

$$\begin{aligned}
[\![N]\!]_u &= [\![M\langle\!\langle U/x\rangle\!\rangle]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
&\longrightarrow^* (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})([\![x[i]]\!]_j \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \qquad (*) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})(\overline{x^!}?(x_k).x_k.l_i;[x_k \leftrightarrow j] \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \qquad (*) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})((\nu x_k)(x_k.l_i;[x_k \leftrightarrow j] \mid [\![U]\!]_{x_k}) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})((\nu x_k)(x_k.l_i;[x_k \leftrightarrow j] \mid x_k.\mathtt{case}_{U_i\in U}\{\mathtt{l}_i : [\![U_i]\!]_x\}) \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})([\![1^!]\!]_j \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu\widetilde{y})(j.\mathtt{none} \mid P_i) \mid !x^!.(x_k).[\![U]\!]_{x_k}) = [\![\mathbb{M}]\!]_u
\end{aligned}$$

and the result follows.

8. **Case** $[\mathtt{RS} : \mathtt{Cons}_1]$**:**
Then, $\mathbb{N} = \mathtt{fail}^{\widetilde{x}}\ C \star U$ and $\mathbb{N} \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x}\uplus\widetilde{y}} = \mathbb{M}$ where $\widetilde{y} = \mathsf{lfv}(C)$. Notice that

$$\begin{aligned}
[\![N]\!]_u &= [\![\mathtt{fail}^{\widetilde{x}}\ C \star U]\!]_u \\
&= \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu v)([\![\mathtt{fail}^{\widetilde{x}}]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)};\overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \\
&= \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu v)(v.\overline{\mathtt{none}} \mid \widetilde{x}.\overline{\mathtt{none}} \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)};\overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \\
&\longrightarrow \bigoplus_{C_i\in\mathsf{PER}(C)} u.\overline{\mathtt{none}} \mid \widetilde{x}.\overline{\mathtt{none}} \mid \widetilde{y}.\overline{\mathtt{none}} = \bigoplus_{C_i\in\mathsf{PER}(C)} u.\overline{\mathtt{none}} \mid \widetilde{x}.\overline{\mathtt{none}} \mid \widetilde{y}.\overline{\mathtt{none}} = [\![\mathbb{M}]\!]_u
\end{aligned}$$

and the result follows.

9. **Cases** $[\mathtt{RS} : \mathtt{Cons}_2]$**,** $[\mathtt{RS} : \mathtt{Cons}_3]$ **and** $[\mathtt{RS} : \mathtt{Cons}_4]$**:**
These cases follow by IH similarly to the previous.

◀

F.2.2 Soundness

▶ **Theorem 67** (Well Formed Weak Operational Soundness)**.** *Let $\mathbb{N}$ be a well-formed, linearly closed $u\widehat{\lambda}^{\lightning}_{\oplus}$ expression. If $[\![\mathbb{N}]\!]_u \longrightarrow^* Q$ then there exist Q' and $\mathbb{N}'$ such that $Q \longrightarrow^* Q'$, $\mathbb{N} \longrightarrow^*_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u \equiv Q'$.*

Proof. By induction on the structure of $\mathbb{N}$ and then induction on the number of reductions of $[\![\mathbb{N}]\!] \longrightarrow^* Q$.

1. **Base case:** $\mathbb{N} = x$, $\mathbb{N} = x[j]$, $\mathbb{N} = \mathtt{fail}^{\emptyset}$ and $\mathbb{N} = \lambda x.(M[\widetilde{x} \leftarrow x])$. .
No reductions can take place, and the result follows trivially. $Q = [\![\mathbb{N}]\!]_u \longrightarrow^0 [\![\mathbb{N}]\!]_u = Q'$ and $x \longrightarrow^0 x = \mathbb{N}'$.
2. $\mathbb{N} = M(C \star U)$.
Then, $[\![M(C \star U)]\!]_u = \bigoplus_{C_i\in\mathsf{PER}(C)}(\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)};\overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x))$, and we are able to perform the reductions from $[\![M(C \star U)]\!]_u$.
We now proceed by induction on k, with $[\![\mathbb{N}]\!]_u \longrightarrow^k Q$. There are two main cases:

a. When $k = 0$ the thesis follows easily:
We have $Q = [\![M(C\star U)]\!]_u \longrightarrow^0 [\![M(C\star U)]\!]_u = Q'$ and $M(C\star U) \longrightarrow^0 M(C\star U) = \mathbb{N}'$.

b. The interesting case is when $k \geq 1$.
Then, for some process R and n, m such that $k = n + m$, we have the following:

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \\
&\longrightarrow^m \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(R \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \longrightarrow^n Q
\end{aligned}$$

Thus, the first $m \geq 0$ reduction steps are internal to $[\![M]\!]_v$; type preservation in $\mathsf{s}\pi$ ensures that, if they occur, these reductions do not discard the possibility of synchronizing with $v.\mathtt{some}$. Then, the first of the $n \geq 0$ reduction steps towards Q is a synchronization between R and $v.\mathtt{some}_{u,\mathsf{lfv}(C)}$.
We consider two sub-cases, depending on the values of m and n:

i. $m = 0$ and $n \geq 1$:
Then $R = [\![\mathbb{M}]\!]_v$ as $[\![\mathbb{M}]\!]_v \longrightarrow^0 [\![\mathbb{M}]\!]_v$. Notice that there are two possibilities of having an unguarded:

A. $M = (\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!|N_1/y_1|\!\rangle \cdots \langle\!|N_p/y_p|\!\rangle [\![U_1/z_1]\!] \cdots [\![U_q/z_q]\!] \quad (p, q \geq 0)$

$$\begin{aligned}
[\![M]\!]_v &= [\![(\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!|N_1/y_1|\!\rangle \cdots \langle\!|N_p/y_p|\!\rangle [\![U_1/z_1]\!] \cdots [\![U_q/z_q]\!]]\!]_v \\
&= (\nu y_1, \cdots, y_p, z_1^!, \cdots, z_q^!)([\![\lambda x.(M'[\widetilde{x} \leftarrow x])]\!]_v \mid y_1.\mathtt{some}_{\mathsf{lfv}(N_1)}; [\![N_1]\!]_{y_1} \mid \cdots \\
&\quad \mid y_p.\mathtt{some}_{\mathsf{lfv}(N_p)}; [\![N_p]\!]_{y_p} \mid !z_1^!.(z_1).[\![U]\!]_{z_1} \mid \cdots \mid !z_q^!.(z_q).[\![U]\!]_{z_q}) \\
&= (\nu \widetilde{y}, \widetilde{z})([\![\lambda x.(M'[\widetilde{x} \leftarrow x])]\!]_v \mid Q'') \\
&= (\nu \widetilde{y}, \widetilde{z})(v.\overline{\mathtt{some}}; v(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_v \mid Q'')
\end{aligned}$$

where $\widetilde{y} = y_1, \cdots, y_p$. $\widetilde{z} = z_1^!, \cdots, z_q^!$ and

$$\begin{aligned}
Q'' = {} & y_1.\mathtt{some}_{\mathsf{lfv}(N_1)}; [\![N_1]\!]_{y_1} \mid \cdots \mid y_p.\mathtt{some}_{\mathsf{lfv}(N_p)}; [\![N_p]\!]_{y_p} \mid !z_1^!.(z_1).[\![U]\!]_{z_1} \mid \cdots \\
& \mid !z_q^!.(z_q).[\![U]\!]_{z_q}.
\end{aligned}$$

With this shape for M, we then have the following:

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![(M\ B)]\!]_u \\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) \\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v, \widetilde{y}, \widetilde{z})(v(x).x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_v \\
&\qquad \mid Q'' \mid \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) && = Q_1 \\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v, \widetilde{y}, \widetilde{z}, x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_v \\
&\qquad \mid Q'' \mid [v \leftrightarrow u] \mid [\![C_i \star U]\!]_x) && = Q_2 \\
&\longrightarrow \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu \widetilde{y}, \widetilde{z}, x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M'[\widetilde{x} \leftarrow x]]\!]_u \mid Q'' \\
&\qquad \mid [\![C_i \star U]\!]_x) && = Q_3
\end{aligned}$$

We also have that

$$\begin{aligned}\mathbb{N} &= (\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!\langle N_1/y_1\rangle\!\rangle \cdots \langle\!\langle N_p/y_p\rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket (C \star U)\\ &\equiv_\lambda (\lambda x.(M'[\widetilde{x} \leftarrow x])(C \star U))\langle\!\langle N_1/y_1\rangle\!\rangle \cdots \langle\!\langle N_p/y_p\rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket\\ &\longrightarrow M'[\widetilde{x} \leftarrow x]\langle\!\langle (C \star U)/x\rangle\!\rangle \langle\!\langle N_1/y_1\rangle\!\rangle \cdots \langle\!\langle N_p/y_p\rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket = \mathbb{M}\end{aligned}$$

Furthermore, we have:

$$\begin{aligned}\llbracket \mathbb{M} \rrbracket_u &= \llbracket M'[\widetilde{x} \leftarrow x]\langle\!\langle (C \star U)/x\rangle\!\rangle \langle\!\langle N_1/y_1\rangle\!\rangle \cdots \langle\!\langle N_p/y_p\rangle\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket \rrbracket_u\\ &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu \widetilde{y}, \widetilde{z}, x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; \llbracket M'[\widetilde{x} \leftarrow x] \rrbracket_u \mid \llbracket C_i \star U \rrbracket_x \mid Q'')\end{aligned}$$

We consider different possibilities for $n \geq 1$; in all the cases, the result follows.

When $n = 1$**:** We have $Q = Q_1$, $\llbracket \mathbb{N} \rrbracket_u \longrightarrow^1 Q_1$. We also have that
- $Q_1 \longrightarrow^2 Q_3 = Q'$,
- $\mathbb{N} \longrightarrow^1 M'[\widetilde{x} \leftarrow x])\langle\!\langle B/x\rangle\!\rangle = \mathbb{N}'$
- and $\llbracket M'[\widetilde{x} \leftarrow x])\langle\!\langle B/x\rangle\!\rangle \rrbracket_u = Q_3$.

When $n = 2$**:** the analysis is similar.

When $n \geq 3$**:** We have $\llbracket \mathbb{N} \rrbracket_u \longrightarrow^3 Q_3 \longrightarrow^l Q$, for $l \geq 0$. We also know that $\mathbb{N} \longrightarrow \mathbb{M}$, $Q_3 = \llbracket \mathbb{M} \rrbracket_u$. By the IH, there exist $Q', \mathbb{N}'$ such that $Q \longrightarrow^i Q'$, $\mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$ and $\llbracket \mathbb{N}' \rrbracket_u = Q'$. Finally, $\llbracket \mathbb{N} \rrbracket_u \longrightarrow^3 Q_3 \longrightarrow^l Q \longrightarrow^i Q'$ and $\mathbb{N} \to \mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$.

B. $M = \mathtt{fail}^{\widetilde{z}}$.

Then, $\llbracket M \rrbracket_v = \llbracket \mathtt{fail}^{\widetilde{z}} \rrbracket_v = v.\overline{\mathtt{none}} \mid \widetilde{z}.\overline{\mathtt{none}}$. With this shape for M, we have:

$$\begin{aligned}\llbracket \mathbb{N} \rrbracket_u &= \llbracket (M\ (C \star U)) \rrbracket_u\\ &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(\llbracket M \rrbracket_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid \llbracket C_i \star U \rrbracket_x))\\ &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(v.\overline{\mathtt{none}} \mid \widetilde{z}.\overline{\mathtt{none}} \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid \llbracket C_i \star U \rrbracket_x))\\ &\longrightarrow \bigoplus_{B_i \in \mathsf{PER}(B)} u.\overline{\mathtt{none}} \mid \widetilde{z}.\overline{\mathtt{none}} \mid \mathsf{lfv}(C_i).\overline{\mathtt{none}}\end{aligned}$$

We also have that $\mathbb{N} = \mathtt{fail}^{\widetilde{x}}\ C \star U \longrightarrow \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{x} \uplus \mathsf{lfv}(C)} = \mathbb{M}$. Furthermore,

$$\begin{aligned}\llbracket \mathbb{M} \rrbracket_u &= \llbracket \sum_{\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z} \uplus \mathsf{lfv}(C)} \rrbracket_u = \bigoplus_{\mathsf{PER}(C)} \llbracket \mathtt{fail}^{\widetilde{z} \uplus \mathsf{lfv}(C)} \rrbracket_u\\ &= \bigoplus_{\mathsf{PER}(C)} u.\overline{\mathtt{none}} \mid \widetilde{z}.\overline{\mathtt{none}} \mid \mathsf{lfv}(C).\overline{\mathtt{none}}.\end{aligned}$$

ii. When $m \geq 1$ and $n \geq 0$, we distinguish two cases:

A. When $n = 0$:

Then, $\bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(R \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid \llbracket C_i \star U \rrbracket_x)) = Q$ and $\llbracket M \rrbracket_u \longrightarrow^m R$ where $m \geq 1$. Then by the IH there exist R' and $\mathbb{M}'$ such that $R \longrightarrow^i R'$, $M \longrightarrow^j_{\equiv_\lambda} \mathbb{M}'$, and $\llbracket \mathbb{M}' \rrbracket_u = R'$. Hence we have that

$$\begin{aligned}\llbracket \mathbb{N} \rrbracket_u &= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(\llbracket M \rrbracket_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid \llbracket C_i \star U \rrbracket_x))\\ &\longrightarrow^m \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(R \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid \llbracket C_i \star U \rrbracket_x)) = Q\end{aligned}$$

We also know that

$$Q \longrightarrow^{i} \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)(R' \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)) = Q'$$

and so the $u\widehat{\lambda}^{\lightning}_{\oplus}$ term can reduce as follows: $\mathbb{N} = (M\ (C \star U)) \longrightarrow^{j}_{\equiv_\lambda} M'\ (C \star U) = \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$.

B. When $n \geq 1$:
Then R has an occurrence of an unguarded $v.\overline{\mathtt{some}}$ or $v.\overline{\mathtt{none}}$, hence it is of the form $[\![(\lambda x.(M'[\widetilde{x} \leftarrow x]))\langle\!| N_1/y_1 |\!\rangle \cdots \langle\!| N_p/y_p |\!\rangle \llbracket U_1/z_1 \rrbracket \cdots \llbracket U_q/z_q \rrbracket]\!]_v$ or $[\![\mathtt{fail}^{\widetilde{x}}]\!]_v$.
This case follows by IH.

This concludes the analysis for the case $\mathbb{N} = (M\ (C \star U))$.

3. $\mathbb{N} = M[\widetilde{x} \leftarrow x]$.
The sharing variable x is not free and the result follows by vacuity.

4. $\mathbb{N} = M[\widetilde{x} \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle$. Then we have

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![M[\widetilde{x} \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle]\!]_u \\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^{\ell}).x(x^{!}).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \mid [\![C_i \star U]\!]_x)
\end{aligned}$$

Let us consider three cases.

a. When $\mathsf{size}(\widetilde{x}) = \mathsf{size}(C)$. Then let us consider the shape of the bag C.

i. When $C = \mathtt{1}$.
We have the following

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= (\nu x)(x.\overline{\mathtt{some}}; x(x^{\ell}).x(x^{!}).x.\mathtt{close}; [\![M[\leftarrow x]]\!]_u \mid [\![\mathtt{1} \star U]\!]_x) \\
&= (\nu x)(x.\overline{\mathtt{some}}; x(x^{\ell}).x(x^{!}).x.\mathtt{close}; [\![M[\leftarrow x]]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(C)}; \overline{x}(x^{\ell}). \\
&\qquad ([\![\mathtt{1}]\!]_{x^{\ell}} \mid \overline{x}(x^{!}).(!x^{!}.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}}))) \\
&\longrightarrow (\nu x)(x(x^{\ell}).x(x^{!}).x.\mathtt{close}; [\![M[\leftarrow x]]\!]_u \mid \overline{x}(x^{\ell}).([\![\mathtt{1}]\!]_{x^{\ell}} \mid \overline{x}(x^{!}). \\
&\qquad (!x^{!}.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}}))) && = Q_1 \\
&\longrightarrow (\nu x, x^{\ell})(x(x^{!}).x.\mathtt{close}; [\![M[\leftarrow x]]\!]_u \mid [\![\mathtt{1}]\!]_{x^{\ell}} \mid \overline{x}(x^{!}). \\
&\qquad (!x^{!}.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}})) && = Q_2 \\
&\longrightarrow (\nu x, x^{\ell}, x^{!})(x.\mathtt{close}; [\![M[\leftarrow x]]\!]_u \mid [\![\mathtt{1}]\!]_{x^{\ell}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i} \mid x.\overline{\mathtt{close}}) && = Q_3 \\
&\longrightarrow (\nu x^{\ell}, x^{!})([\![M[\leftarrow x]]\!]_u \mid [\![\mathtt{1}]\!]_{x^{\ell}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) && = Q_4 \\
&= (\nu x^{\ell}, x^{!})(x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_i).(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}}) \mid \\
&\qquad x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}}) \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) \\
&\longrightarrow (\nu x^{\ell}, x^{!})(\overline{x^{\ell}}(y_i).(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}}) \mid \\
&\qquad x^{\ell}(y_n).(y_n.\overline{\mathtt{some}}; y_n.\overline{\mathtt{close}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}}) \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) && = Q_5 \\
&\longrightarrow (\nu x^{\ell}, x^{!}, y_i)(y_i.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_i.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}} \mid y_i.\overline{\mathtt{some}}; y_i.\overline{\mathtt{close}} \\
&\qquad \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) && = Q_6 \\
&\longrightarrow (\nu x^{\ell}, x^{!}, y_i)(y_i.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}} \mid y_i.\overline{\mathtt{close}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}} \\
&\qquad \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) && = Q_7 \\
&\longrightarrow (\nu x^{\ell}, x^{!})([\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) && = Q_8 \\
&\longrightarrow (\nu x^{!})([\![M]\!]_u \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) = [\![M \llbracket U/x \rrbracket]\!]_u && = Q_9
\end{aligned}$$

Notice how Q_8 has a choice however the x^ℓ name can be closed at any time so for simplicity we only perform communication across this name once all other names have completed their reductions.
Now we proceed by induction on the number of reductions $[\![\mathbb{N}]\!]_u \longrightarrow^k Q$.

A. When $k = 0$, the result follows trivially. Just take $\mathbb{N} = \mathbb{N}'$ and $[\![\mathbb{N}]\!]_u = Q = Q'$.

B. When $k = 1$.
We have $Q = Q_1$, $[\![\mathbb{N}]\!]_u \longrightarrow^1 Q_1$ We also have that $Q_1 \longrightarrow^8 Q_9 = Q'$, $\mathbb{N} \longrightarrow M\langle\!\langle U/x \rangle\!\rangle = M$ and $[\![M]\!]_u = Q_9$

C. When $2 \leq k \leq 8$.
Proceeds similarly to the previous case

D. When $k \geq 9$.
We have $[\![\mathbb{N}]\!]_u \longrightarrow^9 Q_9 \longrightarrow^l Q$, for $l \geq 0$. Since $Q_9 = [\![M]\!]_u$ we apply the induction hypothesis we have that there exist $Q', \mathbb{N}'$ *s.t.* $Q \longrightarrow^i Q', M \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$. Then, $[\![\mathbb{N}]\!]_u \longrightarrow^5 Q_5 \longrightarrow^l Q \longrightarrow^i Q'$ and by the contextual reduction rule it follows that $\mathbb{N} = (M[\leftarrow x])\langle\!\langle 1/x \rangle\!\rangle \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$ and the case holds.

ii. When $C = \wr N_1 \int \cdot \cdots \cdot \wr N_l \int$, for $l \geq 1$. Then,

$$
\begin{aligned}
[\![\mathbb{N}]\!]_u &= [\![M[\widetilde{x} \leftarrow x]\langle\!\langle C \star U/x \rangle\!\rangle]\!]_u\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^\ell).x(x^!).x.\mathtt{close}; [\![M[\widetilde{x} \leftarrow x]]\!]_u \mid [\![C_i \star U]\!]_x)\\
&\longrightarrow^4 \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)([\![M[\widetilde{x} \leftarrow x]]\!]_u \mid [\![C_i]\!]_{x^\ell} \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&= \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!)(x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_1).(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}};\\
&\quad x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_1,\cdots,x_l)}; x^\ell(x_1).\cdots x^\ell.\overline{\mathtt{some}}.\overline{x^\ell}(y_l).(y_l.\mathtt{some}_\emptyset; y_l.\mathtt{close}; 0\\
&\quad \mid x^\ell.\overline{\mathtt{some}}; x^\ell.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_l)}; x^\ell(x_l).x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(y_{l+1}).(y_{l+1}.\mathtt{some}_{u,\mathsf{lfv}(M)};\\
&\quad y_{l+1}.\mathtt{close}; [\![M]\!]_u \mid x^\ell.\overline{\mathtt{none}}))\cdots) \mid x^\ell.\mathtt{some}_{\mathsf{lfv}(C)}; x^\ell(y_1).x^\ell.\mathtt{some}_{y_1,\mathsf{lfv}(C)};\\
&\quad x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_1).(x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid \cdots x^\ell.\mathtt{some}_{\mathsf{lfv}(C_i(l))};\\
&\quad x^\ell(y_l).x^\ell.\mathtt{some}_{y_l,\mathsf{lfv}(C_i(l))}; x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(x_l).(x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l}\\
&\quad \mid y_l.\overline{\mathtt{none}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}})))\\
&\quad \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow^{5l} \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^\ell, x^!, x_1, y_1, \cdots, x_l, y_1)(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid \cdots y_l.\mathtt{some}_\emptyset;\\
&\quad y_l.\mathtt{close}; 0 \mid x^\ell.\overline{\mathtt{some}}; \overline{x^\ell}(y_{l+1}).(y_{l+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{l+1}.\mathtt{close}; [\![M]\!]_u \mid x^\ell.\overline{\mathtt{none}}) \mid\\
&\quad x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid \cdots x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid y_l.\overline{\mathtt{none}} \mid\\
&\quad x^\ell.\mathtt{some}_\emptyset; x^\ell(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \mid x^\ell.\mathtt{some}_\emptyset; x^\ell.\overline{\mathtt{none}}) \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow^5 \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^!, x_1, y_1, \cdots, x_l, y_1)(y_1.\mathtt{some}_\emptyset; y_1.\mathtt{close}; 0 \mid \cdots y_l.\mathtt{some}_\emptyset; y_l.\mathtt{close}; 0\\
&\quad \mid [\![M]\!]_u \mid x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid \cdots x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l}\\
&\quad \mid y_l.\overline{\mathtt{none}} \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow^l \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu x^!, x_1 \cdots, x_l)([\![M]\!]_u \mid x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots\\
&\quad \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid !x^!.(x_i).[\![U]\!]_{x_i})\\
&= [\![\sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(l)/x_l |\!\rangle \lfloor U/x \rceil]\!]_u = Q_{6l+9}
\end{aligned}
$$

The proof follows by induction on the number of reductions $[\![\mathbb{N}]\!]_u \longrightarrow^k Q$.

A. When $k = 0$, the result follows trivially. Just take $\mathbb{N} = \mathbb{N}'$ and $[\![\mathbb{N}]\!]_u = Q = Q'$.

B. When $1 \leq k \leq 6l + 9$.
Let Q_k such that $[\![\mathbb{N}]\!]_u \longrightarrow^k Q_k$. We also have that $Q_k \longrightarrow^{6l+9-k} Q_{6l+9} = Q'$, $\mathbb{N} \longrightarrow^1 \sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(l)/x_l |\!\rangle \lfloor U/x \rceil = \mathbb{N}'$ and $[\![\sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(l)/x_l |\!\rangle \lfloor U/x \rceil]\!]_u = Q_{6l+9}$.

C. When $k > 6l + 9$.
Then, $[\![\mathbb{N}]\!]_u \longrightarrow^{6l+9} Q_{6l+9} \longrightarrow^n Q$ for $n \geq 1$. Also,

$$\mathbb{N} \longrightarrow^1 \sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(l)/x_l |\!\rangle \lfloor U/x \rceil \text{ and}$$

$$Q_{6l+9} = [\![\sum_{C_i \in \mathsf{PER}(C)} M\langle\!| C_i(1)/x_1 |\!\rangle \cdots \langle\!| C_i(l)/x_l |\!\rangle \lfloor U/x \rceil]\!]_u.$$

By the induction hypothesis, there exist Q' and $\mathbb{N}'$ such that $Q \longrightarrow^i Q'$,

$\sum_{C_i\in\mathsf{PER}(C)} M\langle\!| C_i(1)/x_1|\!\rangle\cdots\langle\!| C_i(l)/x_l|\!\rangle\llbracket U/x\rrbracket \longrightarrow^{j}_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$.
Finally, $[\![\mathbb{N}]\!]_u \longrightarrow^{6l+9} Q_{6l+9} \longrightarrow^{n} Q \longrightarrow^{i} Q'$ and

$$\mathbb{N} \to \sum_{C_i\in\mathsf{PER}(C)} M\langle\!| C_i(1)/x_1|\!\rangle\cdots\langle\!| C_i(l)/x_l|\!\rangle\llbracket U/x\rrbracket \longrightarrow^{j}_{\equiv_\lambda} \mathbb{N}'.$$

b. When $\mathsf{size}(\widetilde{x}) > \mathsf{size}(C)$.
Then we have $\mathbb{N} = M[x_1,\cdots,x_k \leftarrow x]\ \langle\!\langle C\star U/x\rangle\!\rangle$ with $C = \wr N_1 \int\cdots\wr N_l\int \quad k>l$.
$\mathbb{N} \longrightarrow \sum_{C_i\in\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}} = \mathbb{M}$ and $\widetilde{z} = (\mathsf{lfv}(M)\setminus\{x_1,\cdots,x_k\})\cup\mathsf{lfv}(C)$. On the one hand, we have: Hence $k = l+m$ for some $m\geq 1$

$$
\begin{aligned}
[\![N]\!]_u &= [\![M[x_1,\cdots,x_k\leftarrow x]\ \langle\!\langle C\star U/x\rangle\!\rangle]\!]_u\\
&= \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x)(x.\overline{\mathtt{some}}; x(x^{\ell}).x(x^{!}).x.\mathtt{close}; [\![M[x_1,\cdots,x_k\leftarrow x]]\!]_u \mid [\![C_i\star U]\!]_x)\\
&\longrightarrow^{4} \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x^{\ell},x^{!})([\![M[x_1,\cdots,x_k\leftarrow x]]\!]_u \mid [\![C_i]\!]_{x^{\ell}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i})\\
&= \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x^{\ell},x^{!})(x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_1).(y_1.\mathtt{some}_{\emptyset}; y_1.\mathtt{close};0 \mid x^{\ell}.\overline{\mathtt{some}}; x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_1,\cdots,x_k)};\\
&\quad x^{\ell}(x_1).\cdots x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_k).(y_k.\mathtt{some}_{\emptyset}; y_k.\mathtt{close};0 \mid x^{\ell}.\overline{\mathtt{some}}; x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_k)}; x^{\ell}(x_k).\\
&\quad x^{\ell}.\overline{\mathtt{some}}; \overline{x^{\ell}}(y_{k+1}).(y_{k+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{k+1}.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}}))\cdots) \mid\\
&\quad x^{\ell}.\mathtt{some}_{\mathsf{lfv}(C)}; x^{\ell}(y_1).x^{\ell}.\mathtt{some}_{y_1,\mathsf{lfv}(C)}; x^{\ell}.\overline{\mathtt{some}}; \overline{x^{\ell}}(x_1).(x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid y_1.\overline{\mathtt{none}} \mid \cdots\\
&\quad x^{\ell}.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; x^{\ell}(y_l).x^{\ell}.\mathtt{some}_{y_l,\mathsf{lfv}(C_i(l))}; x^{\ell}.\overline{\mathtt{some}}; \overline{x^{\ell}}(x_l).(x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid y_l.\overline{\mathtt{none}} \mid\\
&\quad x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}}))) \mid !x^{!}.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow^{5l} \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x^{\ell},x^{!},y_1,x_1,\cdots y_l,x_l)(y_1.\mathtt{some}_{\emptyset}; y_1.\mathtt{close};0 \mid \cdots \mid y_l.\mathtt{some}_{\emptyset}; y_l.\mathtt{close};0\\
&\quad x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_{l+1}).(y_{l+1}.\mathtt{some}_{\emptyset}; y_{l+1}.\mathtt{close};0 \mid x^{\ell}.\overline{\mathtt{some}}; x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_{l+1},\cdots,x_k)}; x^{\ell}(x_{l+1}).\cdots\\
&\quad x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_k).(y_k.\mathtt{some}_{\emptyset}; y_k.\mathtt{close};0 \mid x^{\ell}.\overline{\mathtt{some}}; x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_k)}; x^{\ell}(x_k).\\
&\quad x^{\ell}.\overline{\mathtt{some}}; \overline{x^{\ell}}(y_{k+1}).(y_{k+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{k+1}.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}}))\cdots) \mid\\
&\quad x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid y_1.\overline{\mathtt{none}} \mid \cdots \mid y_l.\overline{\mathtt{none}}\\
&\quad x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}(y_{l+1}).(y_{l+1}.\overline{\mathtt{some}}; y_{l+1}.\overline{\mathtt{close}} \mid x^{\ell}.\mathtt{some}_{\emptyset}; x^{\ell}.\overline{\mathtt{none}}) \mid !x^{!}.(x_i).[\![U]\!]_{x_i}) \qquad (:= P_{\mathbb{N}})
\end{aligned}
$$

$$
\begin{aligned}
P_{\mathbb{N}} &\longrightarrow^{l+5} \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x^{\ell},x^{!},x_1,\cdots,x_l)(x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_{l+1},\cdots,x_k)}; x^{\ell}(x_{l+1}).\cdots\\
&\quad x^{\ell}.\overline{\mathtt{some}}.\overline{x^{\ell}}(y_k).(y_k.\mathtt{some}_{\emptyset}; y_k.\mathtt{close};0 \mid x^{\ell}.\overline{\mathtt{some}}; x^{\ell}.\mathtt{some}_{u,(\mathsf{lfv}(M)\setminus x_k)}; x^{\ell}(x_k).\\
&\quad x^{\ell}.\overline{\mathtt{some}}; \overline{x^{\ell}}(y_{k+1}).(y_{k+1}.\mathtt{some}_{u,\mathsf{lfv}(M)}; y_{k+1}.\mathtt{close}; [\![M]\!]_u \mid x^{\ell}.\overline{\mathtt{none}})) \mid\\
&\quad x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l} \mid x^{\ell}.\overline{\mathtt{none}} \mid !x^{!}.(x_i).[\![U]\!]_{x_i})\\
&\longrightarrow \bigoplus_{C_i\in\mathsf{PER}(C)} (\nu x^{!},x_1,\cdots,x_l)(u.\overline{\mathtt{none}} \mid x_1.\overline{\mathtt{none}} \mid \cdots \mid x_l.\overline{\mathtt{none}} \mid (\mathsf{lfv}(M)\setminus\{x_1,\cdots,x_k\}).\overline{\mathtt{none}} \mid\\
&\quad x_1.\mathtt{some}_{\mathsf{lfv}(C_i(1))}; [\![C_i(1)]\!]_{x_1} \mid \cdots \mid x_l.\mathtt{some}_{\mathsf{lfv}(C_i(l))}; [\![C_i(l)]\!]_{x_l})\\
&\longrightarrow^{l} \bigoplus_{C_i\in\mathsf{PER}(C)} u.\overline{\mathtt{none}} \mid (\mathsf{lfv}(M)\setminus\{x_1,\cdots,x_k\}).\overline{\mathtt{none}} \mid \mathsf{lfv}(C).\overline{\mathtt{none}}\\
&= [\![\sum_{C_i\in\mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}}]\!]_u = Q_{7l+10}
\end{aligned}
$$

The rest of the proof is by induction on the number of reductions $[\![\mathbb{N}]\!]_u \longrightarrow^j Q$.

i. When $j = 0$, the result follows trivially. Just take $\mathbb{N} = \mathbb{N}'$ and $[\![\mathbb{N}]\!]_u = Q = Q'$.

ii. When $1 \leq j \leq 7l + 10$.
Let Q_j be such that $[\![\mathbb{N}]\!]_u \longrightarrow^j Q_j$. By the steps above one has $Q_j \longrightarrow^{7l+10-j} Q_{7l+6} = Q'$,
$\mathbb{N} \longrightarrow^1 \sum_{C_i \in \mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}} = \mathbb{N}'$; and $[\![\sum_{C_i \in \mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}}]\!]_u = Q_{7l+10}$.

iii. When $j > 7l + 10$.
In this case, we have

$$[\![\mathbb{N}]\!]_u \longrightarrow^{7l+10} Q_{7l+10} \longrightarrow^n Q,$$

for $n \geq 1$. We also know that $\mathbb{N} \longrightarrow^1 \sum_{C_i \in \mathsf{PER}(C)} \mathtt{fail}^{\widetilde{z}}$. However no further reductions can be performed.

c. When $\mathsf{size}(\widetilde{x}) < \mathsf{size}(C)$, the proof proceeds similarly to the previous case.

5. $\mathbb{N} = M\langle\!|N'/x|\!\rangle$.
In this case, $[\![M\langle\!|N'/x|\!\rangle]\!]_u = (\nu x)([\![M]\!]_u \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)$. Therefore,

$$[\![\mathbb{N}]\!]_u = (\nu x)([\![M]\!]_u \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) \longrightarrow^m (\nu x)(R \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) \longrightarrow^n Q,$$

for some process R. Where $\longrightarrow^n$ is a reduction that initially synchronizes with $x.\mathsf{some}_{\mathsf{lfv}(N')}$ when $n \geq 1$, $n + m = k \geq 1$. Type preservation in $\mathsf{s}\pi$ ensures reducing $[\![M]\!]_v \longrightarrow^m$ does not consume possible synchronizations with $x.\mathtt{some}$, if they occur. Let us consider the the possible sizes of both m and n.

a. For $m = 0$ and $n \geq 1$.
We have that $R = [\![M]\!]_u$ as $[\![M]\!]_u \longrightarrow^0 [\![M]\!]_u$.
Notice that there are two possibilities of having an unguarded $x.\overline{\mathtt{some}}$ or $x.\overline{\mathtt{none}}$ without internal reductions:

i. $M = \mathtt{fail}^{x,\widetilde{y}}$.

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= (\nu x)([\![M]\!]_u \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)\\
&= (\nu x)([\![\mathtt{fail}^{x,\widetilde{y}}]\!]_u \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)\\
&= (\nu x)(u.\overline{\mathtt{none}} \mid x.\overline{\mathtt{none}} \mid \widetilde{y}.\overline{\mathtt{none}} \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)\\
&\longrightarrow u.\overline{\mathtt{none}} \mid \widetilde{y}.\overline{\mathtt{none}} \mid \mathsf{lfv}(N').\overline{\mathtt{none}}
\end{aligned}$$

Notice that no further reductions can be performed. Thus,

$$[\![\mathbb{N}]\!]_u \longrightarrow u.\overline{\mathtt{none}} \mid \widetilde{y}.\overline{\mathtt{none}} \mid \mathsf{fv}(N').\overline{\mathtt{none}} = Q'.$$

We also have that $\mathbb{N} \longrightarrow \mathtt{fail}^{\widetilde{y} \cup \mathsf{lfv}(N')} = \mathbb{N}'$ and $[\![\mathtt{fail}^{\widetilde{y} \cup \mathsf{lfv}(N')}]\!]_u = Q'$.

ii. $\mathsf{head}(M) = x$.
By the diamond property we will be reducing each non-deterministic choice of a process simultaneously. Then we have the following

$$\begin{aligned}
[\![\mathbb{N}]\!]_u &= (\nu x)(\bigoplus_{i \in I} (\nu \widetilde{y})([\![x]\!]_j \mid P_i) \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)\\
&= (\nu x)(\bigoplus_{i \in I} (\nu \widetilde{y})(x.\overline{\mathtt{some}}; [x \leftrightarrow j] \mid P_i) \mid x.\mathsf{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x)\\
&\longrightarrow (\nu x)(\bigoplus_{i \in I} (\nu \widetilde{y})([x \leftrightarrow j] \mid P_i) \mid [\![N']\!]_x) & = Q_1\\
&\longrightarrow \bigoplus_{i \in I} (\nu \widetilde{y})([\![N']\!]_j \mid P_i) & = Q_2
\end{aligned}$$

In addition, $\mathbb{N} = M\langle\!| N'/x |\!\rangle \longrightarrow M\{\!| N'/x |\!\} = \mathbb{M}$. Finally,

$$[\![\mathbb{M}]\!]_u = [\![M\{\!| N'/x |\!\}]\!]_u = \bigoplus_{i \in I} (\nu \widetilde{y})([\![N']\!]_j \mid P_i) \quad = Q_2.$$

A. When $n = 1$:
Then, $Q = Q_1$ and $[\![\mathbb{N}]\!]_u \longrightarrow^1 Q_1$. Also, $Q_1 \longrightarrow^1 Q_2 = Q'$, $\mathbb{N} \longrightarrow^1 M\{\!| N'/x |\!\} = \mathbb{N}'$ and $[\![M\{\!| N'/x |\!\}]\!]_u = Q_2$.

B. When $n \geq 2$:
Then $[\![\mathbb{N}]\!]_u \longrightarrow^2 Q_2 \longrightarrow^l Q$, for $l \geq 0$. Also, $\mathbb{N} \to \mathbb{M}$, $Q_2 = [\![\mathbb{M}]\!]_u$. By the induction hypothesis, there exist Q' and $\mathbb{N}'$ such that $Q \longrightarrow^i Q'$, $\mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$. Finally, $[\![\mathbb{N}]\!]_u \longrightarrow^2 Q_2 \longrightarrow^l Q \longrightarrow^i Q'$ and $\mathbb{N} \to \mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$.

b. For $m \geq 1$ and $n \geq 0$.

i. When $n = 0$.
Then $(\nu x)(R \mid x.\mathtt{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) = Q$ and $[\![M]\!]_u \longrightarrow^m R$ where $m \geq 1$. By the IH there exist R' and $\mathbb{M}'$ such that $R \longrightarrow^i R'$, $M \longrightarrow^j_{\equiv_\lambda} \mathbb{M}'$ and $[\![\mathbb{M}']\!]_u = R'$. Thus,

$$[\![\mathbb{N}]\!]_u = (\nu x)([\![M]\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) \longrightarrow^m (\nu x)(R \mid x.\mathtt{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) = Q$$

Also, $Q \longrightarrow^i (\nu x)(R' \mid x.\mathtt{some}_{\mathsf{lfv}(N')}; [\![N']\!]_x) = Q'$, and the term can reduce as follows: $\mathbb{N} = M\langle\!| N'/x |\!\rangle \longrightarrow^j_{\equiv_\lambda} \sum_{M'_i \in \mathbb{M}'} M'_i\langle\!| N'/x |\!\rangle = \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$

ii. When $n \geq 1$. Then R has an occurrence of an unguarded $x.\overline{\mathtt{some}}$ or $x.\overline{\mathtt{none}}$, this case follows by IH.

6. $\mathbb{N} = M\lfloor\!\lfloor U/x \rceil\!\rceil$.
In this case, $[\![M\lfloor\!\lfloor U/x \rceil\!\rceil]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i})$. Then,

$$[\![\mathbb{N}]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i}) \longrightarrow^m (\nu x^!)(R \mid !x^!.(x_i).[\![U]\!]_{x_i}) \longrightarrow^n Q.$$

for some process R. Where $\longrightarrow^n$ is a reduction initially synchronises with $!x^!.(x_i)$ when $n \geq 1$, $n + m = k \geq 1$. Type preservation in $\mathsf{s}\pi$ ensures reducing $[\![M]\!]_v \longrightarrow^m$ doesn't consume possible synchronisations with $!x^!.(x_i)$ if they occur. Let us consider the the possible sizes of both m and n.

a. For $m = 0$ and $n \geq 1$.
In this case, $R = [\![M]\!]_u$ as $[\![M]\!]_u \longrightarrow^0 [\![M]\!]_u$.

Notice that the only possibility of having an unguarded $\overline{x^!}?(x_i)$ without internal reductions is when $\mathsf{head}(M) = x[ind]$. By the diamond property we will be reducing each non-deterministic choice of a process simultaneously. Then we have the following:

$$
\begin{aligned}
[\![\mathbb{N}]\!]_u &= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![x[ind]]\!]_j \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})(\overline{x^!}?(x_i).x_i.l_{ind};[x_i \leftrightarrow j] \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})((\nu x_i)(x_i.l_{ind};[x_i \leftrightarrow j] \mid [\![U]\!]_{x_i}) \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) &= Q_1 \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})((\nu x_i)(x_i.l_{ind};[x_i \leftrightarrow j] \mid x_i.case(ind.[\![U_{ind}]\!]_{x_i})) \mid P_i) \\
&\mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})((\nu x_i)([x_i \leftrightarrow j] \mid [\![U_{ind}]\!]_{x_i}) \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) &= Q_2 \\
&\longrightarrow (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![U_{ind}]\!]_j \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) &= Q_3
\end{aligned}
$$

We consider the two cases of the form of U_{ind} and show that the choice of U_{ind} is inconsequential

- When $U_i = \wr N \int^!$:
  In this case, $\mathbb{N} = M\lfloor\!\lfloor U/x \rceil\!\rceil \longrightarrow M\{\!|N/x^!|\!\}\lfloor\!\lfloor U/x \rceil\!\rceil = \mathbb{M}$. and

$$
[\![\mathbb{M}]\!]_u = [\![M\{\!|N/x^!|\!\}\lfloor\!\lfloor U/x \rceil\!\rceil]\!]_u = (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![\wr N \int]\!]_j \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \quad = Q_3
$$

- When $U_i = \mathtt{1}^!$:
  In this case, $\mathbb{N} = M\lfloor\!\lfloor U/x \rceil\!\rceil \longrightarrow M\{\!|\mathtt{fail}^{\emptyset}/x^!|\!\}\lfloor\!\lfloor U/x \rceil\!\rceil = \mathbb{M}$.
  Notice that $[\![\mathtt{1}^!]\!]_j = j.\mathtt{none}$ and that $[\![\mathtt{fail}^{\emptyset}]\!]_j = j.\overline{\mathtt{none}}$. In addition,

$$
\begin{aligned}
[\![\mathbb{M}]\!]_u &= [\![M\{\!|\mathtt{fail}^{\emptyset}/x^!|\!\}\lfloor\!\lfloor U/x \rceil\!\rceil]\!]_u \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![\mathtt{fail}^{\emptyset}]\!]_j \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) \\
&= (\nu x^!)(\bigoplus_{i\in I}(\nu \widetilde{y})([\![\mathtt{1}^!]\!]_j \mid P_i) \mid !x^!.(x_i).[\![U]\!]_{x_i}) &= Q_3
\end{aligned}
$$

Both choices give an $\mathbb{M}$ that are equivalent to Q_3.

i. When $n \leq 2$.
In this case, $Q = Q_n$ and $[\![\mathbb{N}]\!]_u \longrightarrow^n Q_n$.
Also, $Q_n \longrightarrow^{3-n} Q_3 = Q'$, $\mathbb{N} \longrightarrow^1 \mathbb{M} = \mathbb{N}'$ and $[\![\mathbb{M}]\!]_u = Q_2$.

ii. When $n \geq 3$.
We have $[\![\mathbb{N}]\!]_u \longrightarrow^3 Q_3 \longrightarrow^l Q$ for $l \geq 0$. We also know that $\mathbb{N} \to \mathbb{M}$, $Q_3 = [\![\mathbb{M}]\!]_u$.
By the IH, there exist Q and $\mathbb{N}'$ such that $Q \longrightarrow^i Q'$, $\mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$.
Finally, $[\![\mathbb{N}]\!]_u \longrightarrow^2 Q_3 \longrightarrow^l Q \longrightarrow^i Q'$ and $\mathbb{N} \to \mathbb{M} \longrightarrow^j_{\equiv_\lambda} \mathbb{N}'$.

b. For $m \geq 1$ and $n \geq 0$.

i. When $n = 0$.
Then $(\nu x^!)(R \mid !x^!.(x_i).[\![U]\!]_{x_i}) = Q$ and $[\![M]\!]_u \longrightarrow^m R$ where $m \geq 1$. By the IH there exist R' and $\mathbb{M}'$ such that $R \longrightarrow^i R'$, $M \longrightarrow^j_{\equiv_\lambda} \mathbb{M}'$ and $[\![\mathbb{M}']\!]_u = R'$. Hence,

$$
[\![\mathbb{N}]\!]_u = (\nu x^!)([\![M]\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i}) \longrightarrow^m (\nu x^!)(R \mid !x^!.(x_i).[\![U]\!]_{x_i}) = Q.
$$

In addition, $Q \longrightarrow^i (\nu x^!)(R' \mid !x^!.(x_i).[\![U]\!]_{x_i}) = Q$, and the term can reduce as follows: $\mathbb{N} = M\lfloor\!\lfloor U/x \rceil\!\rceil \longrightarrow^j_{\equiv_\lambda} \sum_{M_i' \in \mathbb{M}'} M_i'\lfloor\!\lfloor U/x \rceil\!\rceil = \mathbb{N}'$ and $[\![\mathbb{N}']\!]_u = Q'$.

ii. When $n \geq 1$.
Then R has an occurrence of an unguarded $\overline{x^!}?(x_i)$, and the case follows by IH.

◀

F.3 Success Sensitiveness of $[\![\cdot]\!]_u$

We say that a process occurs *guarded* when it occurs behind a prefix (input, output, closing of channels, servers, server request, choice an selection and non-deterministic session behaviour). Formally,

▶ **Definition 68.** *A process* $P \in \mathsf{s}\pi$ *is* guarded *if* $\alpha.P$, $\alpha; P$ *or* $x.\mathtt{case}_{i\in I}\{\mathtt{l}_i : P\}$*, where* $\alpha = \overline{x}(y), x(y), x.\overline{\mathtt{close}}, x.\mathtt{close}, x.\overline{\mathtt{some}}, x.\mathtt{some}_{(w_1,\cdots,w_n)}, x.\mathtt{l}_i, !x(y), \overline{x}?(y)$. *We say it occurs* unguarded *if it is not guarded for any prefix.*

▶ **Proposition 69** (Preservation of Success)**.** *For all* $M \in u\widehat{\lambda}^{\sharp}_{\oplus}$*, the following hold:*
1. $\mathsf{head}(M) = \checkmark \implies [\![M]\!] = P \mid \checkmark \oplus Q$
2. $[\![M]\!]_u = P \mid \checkmark \oplus Q \implies \mathsf{head}(M) = \checkmark$

Proof. Proof of both cases by induction on the structure of M.

1. We only need to consider terms of the following form:

   a. $M = \checkmark$:
   This case is immediate.
   b. $M = N\ (C \star U)$:
   Then, $\mathsf{head}(N\ (C \star U)) = \mathsf{head}(N)$. If $\mathsf{head}(N) = \checkmark$, then

$$[\![M(C \star U)]\!]_u = \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![M]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x)).$$

   By the IH, $\checkmark$ is unguarded in $[\![N]\!]_u$.
   c. $M = M'\langle\!| N/x |\!\rangle$
   Then we have that $\mathsf{head}(M'\langle\!| N/x |\!\rangle) = \mathsf{head}(M') = \checkmark$. Then $[\![M'\langle\!| N/x |\!\rangle]\!]_u = (\nu x)([\![M']\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x$
   and by the IH $\checkmark$ is unguarded in $[\![M']\!]_u$.
   d. $M = M'\lfloor\!\lfloor U/x \rceil\!\rceil$
   Then we have that $\mathsf{head}(M'\lfloor\!\lfloor U/x \rceil\!\rceil) = \mathsf{head}(M') = \checkmark$. Then $[\![M'\lfloor\!\lfloor U/x \rceil\!\rceil]\!]_u = (\nu x^!)([\![M']\!]_u \mid !x^!.(x_i).[\![U]\!]_{x_i})$
   and by the IH $\checkmark$ is unguarded in $[\![M']\!]_u$.

2. We only need to consider terms of the following form:

   a. Case $M = \checkmark$**:**
   Then, $[\![\checkmark]\!]_u = \checkmark$ which is an unguarded occurrence of $\checkmark$ and that $\mathsf{head}(\checkmark) = \checkmark$.
   b. Case $M = N(C \star U)$**:**
   Then, $[\![N(C\star U)]\!]_u = \bigoplus_{C_i \in \mathsf{PER}(C)} (\nu v)([\![N]\!]_v \mid v.\mathtt{some}_{u,\mathsf{lfv}(C)}; \overline{v}(x).([v \leftrightarrow u] \mid [\![C_i \star U]\!]_x))$. The only occurrence of an unguarded $\checkmark$ can occur is within $[\![N]\!]_v$. By the IH, $\mathsf{head}(N) = \checkmark$ and finally $\mathsf{head}(N\ B) = \mathsf{head}(N)$.
   c. Case $M = M'\langle\!| N/x |\!\rangle$**:**
   Then, $[\![M'\langle\!| N/x |\!\rangle]\!]_u = (\nu x)([\![M']\!]_u \mid x.\mathtt{some}_{\mathsf{lfv}(N)}; [\![N]\!]_x)$, an unguarded occurrence of $\checkmark$ can only occur within $[\![M']\!]_u$. By the IH, $\mathsf{head}(M') = \checkmark$ and hence $\mathsf{head}(M'\langle\!| N/x |\!\rangle) = \mathsf{head}(M')$.

d. Case $M = M'\llbracket U/x \rrbracket$**:** This case is analogous to the previous.

◀

▶ **Theorem 70** (Success Sensitivity). *The encoding* $\llbracket - \rrbracket_u : u\widehat{\lambda}^{\lightning}_{\oplus} \to \mathsf{s}\pi$ *is success sensitive on well formed linearly closed expression if for any expression we have* $\mathbb{M} \Downarrow_{\checkmark}$ *iff* $\llbracket \mathbb{M} \rrbracket_u \Downarrow_{\checkmark}$.

Proof. We proceed with the proof in two parts.

1. Suppose that $\mathbb{M} \Downarrow_{\checkmark}$. We will prove that $\llbracket \mathbb{M} \rrbracket \Downarrow_{\checkmark}$.
   By Def. 59, there exists $\mathbb{M}' = M_1, \cdots, M_k$ such that $\mathbb{M} \longrightarrow^* \mathbb{M}'$ and $\mathsf{head}(M_j') = \checkmark$, for some $j \in \{1, \ldots, k\}$ and term M_j' such that $M_j \equiv_\lambda M_j'$. By completeness, there exists Q such that $\llbracket \mathbb{M} \rrbracket_u \longrightarrow^* Q = \llbracket \mathbb{M}' \rrbracket_u$.
   We wish to show that there exists Q'such that $Q \longrightarrow^* Q'$ and Q' has an unguarded occurrence of $\checkmark$.
   From $Q = \llbracket \mathbb{M}' \rrbracket_u$ and due to compositionality and the homomorphic preservation of non-determinism we have that $Q = \llbracket M_1 \rrbracket_u \oplus \cdots \oplus \llbracket M_k \rrbracket_u$.
   By Proposition 69 (1) we have that $\mathsf{head}(M_j) = \checkmark \implies \llbracket M_j \rrbracket_u = P \mid \checkmark \oplus Q'$. Hence Q reduces to a process that has an unguarded occurence of $\checkmark$.
2. Suppose that $\llbracket \mathbb{M} \rrbracket_u \Downarrow_{\checkmark}$. We will prove that $\mathbb{M} \Downarrow_{\checkmark}$.
   By operational soundness (Lemma 67) we have that if $\llbracket \mathbb{N} \rrbracket_u \longrightarrow^* Q$ then there exist Q' and $\mathbb{N}'$ such that $Q \longrightarrow^* Q'$, $\mathbb{N} \longrightarrow^*_{\equiv_\lambda} \mathbb{N}'$ and $\llbracket \mathbb{N}' \rrbracket_u = Q'$.
   Since $\llbracket \mathbb{M} \rrbracket_u \longrightarrow^* P_1 \oplus \ldots \oplus P_k$, and $P_j' = P_j'' \mid \checkmark$, for some j and P_j', such that $P_j \equiv P_j'$.
   Notice that if $\llbracket \mathbb{M} \rrbracket_u$ is itself a term with unguarded $\checkmark$, say $\llbracket \mathbb{M} \rrbracket_u = P \mid \checkmark$, then $\mathbb{M}$ is itself headed with $\checkmark$, from Proposition 69 (2).
   In the case $\llbracket \mathbb{M} \rrbracket_u = P_1 \oplus \ldots \oplus P_k$, $k \geq 2$, and $\checkmark$ occurs unguarded in an P_j, The encoding acts homomorphically over sums and the reasoning is similar. We have that $P_j = P_j' \mid \checkmark$ we apply Proposition 69 (2).

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